Revenue and marginal revenue:

Likewise, if $r = g(q)$ is a revenue function, i.e., r is the revenue generated by selling q units of output, then the *marginal revenue* is the additional revenue generated by selling one more unit. I.e.,

$$MR = r(q_0 + 1) - r(q_0) = \Delta r \approx \left.\frac{dr}{dq}\right|_{q=q_0} \cdot \Delta q = \left.\frac{dr}{dq}\right|_{q=q_0},$$

because once again, $\Delta q = 1$.

The derivative $\dfrac{dr}{dq}$ is called the *marginal revenue* function.

Example. Find the marginal revenue function for a firm whose *demand equation* is given by $p = 100 - \sqrt{q}$, where p is the price of the firm's good.

First, find the revenue function (as a function of q):

$$r = pq = (100 - \sqrt{q})q = 100q - q^{3/2}.$$

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Then differentiate to find the marginal revenue function:

$$\frac{dr}{dq} = 100 - \frac{3}{2}q^{1/2}.$$

General terminology: In the context of applying differential calculus to economics, the word *marginal* (or a phrase containing the word marginal) means derivative.

E.g., as we have just seen, the *marginal* revenue function is the *derivative* of the revenue function and the *marginal* cost function is the *derivative* of the cost function.

More examples:

(*) If $q = f(l)$ gives a firm's output (q) as a function of its labor input (l), then the derivative, dq/dl is called the *marginal product* (of labor).

(*) If a nation's annual consumption (spending) (C) is given as a function of the annual national income (Y), then the derivative dC/dY is called the *marginal propensity* to consume.

(*) A firm's revenue (r) depends on its output (q), and the firm's output depends on its labor input (l). This means that the firm's revenue can be expressed as a function of its labor input, $r = h(l)$. The derivative of this function, dr/dl, is called the firm's *marginal revenue product*.

Example. Suppose that the *marginal propensity to consume* of a small nation is given by

$$\frac{dC}{dY} = \frac{9Y + 10}{10Y + 1},$$

where the nation's income Y and consumption $C = f(Y)$ are both measured in billions of dollars.

The nation's current income is \$8 billion. By approximately how much will consumption increase if income increases by \$400 *million.*

First, observe that $\Delta Y = \dfrac{400,000,000}{1,000,000,000} = 0.4$, because of the units of measurement.

Now use linear approximation:

$$\Delta C \approx \left.\frac{dC}{dY}\right|_{Y=8} \cdot \Delta Y = \frac{9 \cdot 8 + 10}{10 \cdot 8 + 1} \cdot 0.4 \approx 0.405$$

Interpretation: Based on this model, if national income increases by \$400 million from its current level, national consumption will increase by about \$405 million (so the nation will incur about \$5 million in debt).

The product rule.

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 1. $\dfrac{d}{dx}\big((x^2 + 4x + 5)(5x + 3)\big) = \dots$

$$\begin{aligned}
&= \left(\frac{d}{dx}(x^2 + 4x + 5)\right)(5x + 3) + (x^2 + 4x + 5)\left(\frac{d}{dx}(5x + 3)\right) \\
&= (2x + 4)(5x + 3) + 5(x^2 + 4x + 5) \\
&= 10x^2 + 26x + 12 + 5x^2 + 20x + 25 \\
&= 15x^2 + 46x + 37
\end{aligned}$$

Check: $(x^2 + 4x + 5)(5x + 3) = 5x^3 + 23x^2 + 37x + 15$, so

$\dfrac{d}{dx}\big((x^2 + 4x + 5)(5x + 3)\big) = \dfrac{d}{dx}\big(5x^3 + 23x^2 + 37x + 15\big) = \dots$ ✓

Example 2. Find the derivative of $y = \dfrac{3x^2 + 2x + 1}{x^2 + 2}$.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$\begin{aligned}
y' &= \frac{\left(3x^2 + 2x + 1\right)' \cdot \left(x^2 + 2\right) - \left(3x^2 + 2x + 1\right) \cdot \left(x^2 + 2\right)'}{\left(x^2 + 2\right)^2} \\
&= \frac{(6x + 2) \cdot \left(x^2 + 2\right) - \left(3x^2 + 2x + 1\right) \cdot 2x}{\left(x^2 + 2\right)^2} \\
&= \frac{\left(6x^3 + 2x^2 + 12x + 4\right) - \left(6x^3 + 4x^2 + 2x\right)}{\left(x^2 + 2\right)^2} \\
&= \frac{-2x^2 + 10x + 4}{\left(x^2 + 2\right)^2}
\end{aligned}$$

Example 3. Find the interval(s) where the slope of $s = \dfrac{3t}{t^2+1}$ is positive.

The slope of this graph is positive at the points t where $ds/dt > 0$, and...

$$\frac{ds}{dt} = \left(\frac{3t}{t^2+1}\right)' = \frac{(3t)'(t^2+1) - 3t(t^2+1)'}{(t^2+1)^2}$$
$$= \frac{3(t^2+1) - 3t \cdot 2t}{(t^2+1)^2}$$
$$= \frac{3-3t^2}{(t^2+1)^2} = \frac{3(1-t^2)}{(t^2+1)^2}.$$

Observation: $3 > 0$ and $(t^2+1)^2 > 0$ for all t.

Therefore $ds/dt > 0$ when $1 - t^2 > 0$, i.e., the slope is positive when $-1 < t < 1$.

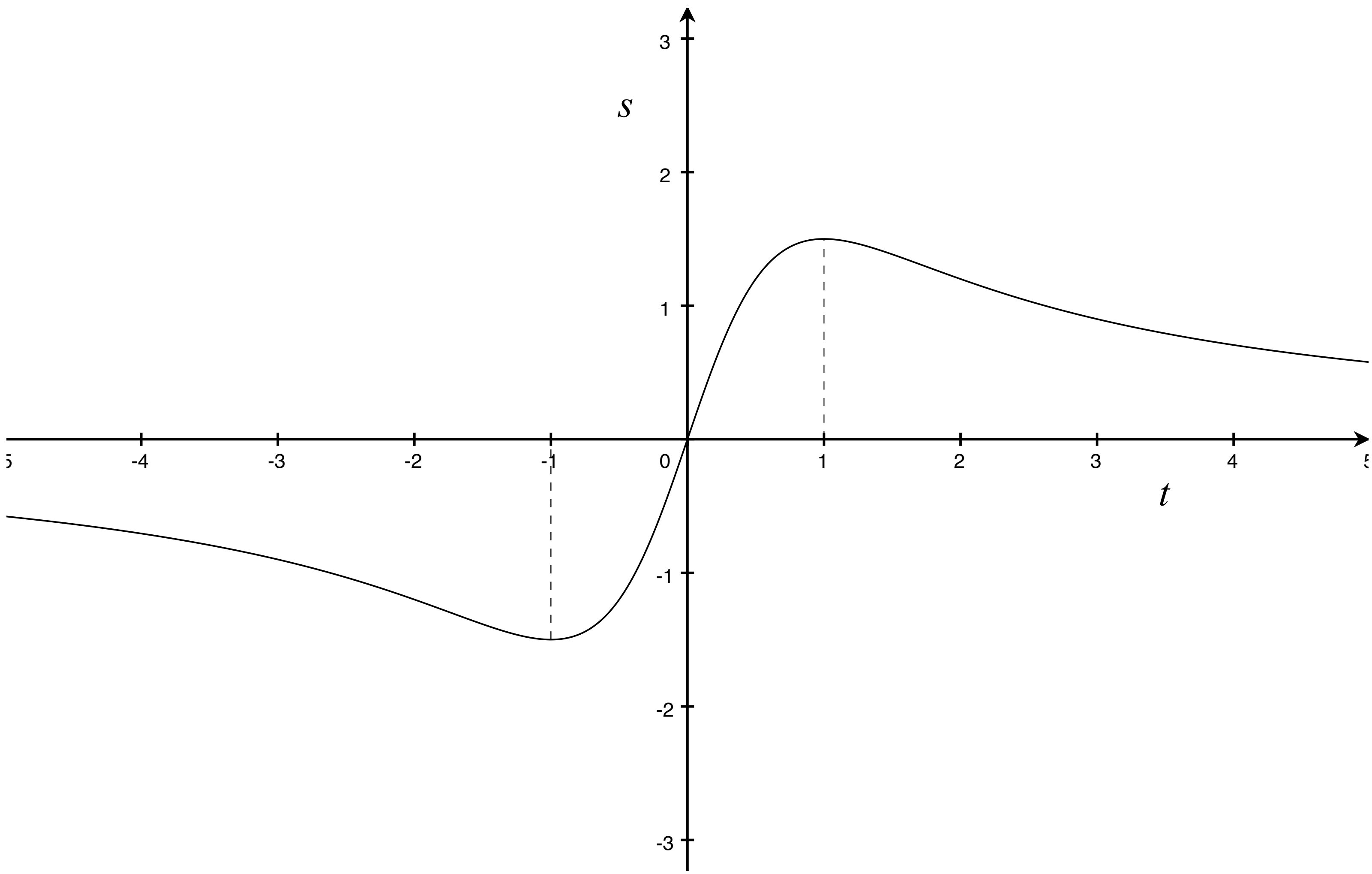

Figure 1: The graph of $s = \dfrac{3t}{t^2+1}$.

Example 4. Find the derivative of $g(x) = (2x+1)(x+3)(3x+5)$

$$\begin{aligned} g'(x) &= (2x+1)' \big[(x+3)(3x+5)\big] + (2x+1)\big[(x+3)(3x+5)\big]' \\ &= 2(x+3)(3x+5) + (2x+1)\big[(x+3)'(3x+5) + (x+3)(3x+5)'\big] \\ &= 2(x+3)(3x+5) + (2x+1)(3x+5) + 3(2x+1)(x+3) \\ &= (6x^2+28x+30) + (6x^2+13x+5) + (6x^2+27x+9) \\ &= 18x^2 + 68x + 44 \end{aligned}$$

More generally:

$$\frac{d}{dx}\big(f(x)g(x)h(x)\big) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

and

$$\begin{aligned} \frac{d}{dx}\big(f(x)g(x)h(x)j(x)\big) = f'(x)g(x)h(x)j(x) + f(x)g'(x)h(x)j(x) \\ + f(x)g(x)h'(x)j(x) + f(x)g(x)h(x)j'(x) \end{aligned}$$

etc.

Example 5. Find the derivative of $y = \dfrac{x^3 - 5x + 7}{2x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x^2 - 5)2x^2 - 4x(x^3 - 5x + 7)}{(2x^2)^2} \\ &= \frac{6x^4 - 10x^2 - 4x^4 + 20x^2 - 28x}{4x^4} \\ &= \frac{2x^4 + 10x^2 - 28x}{4x^4} = \frac{x^4 + 5x^2 - 14x}{2x^4} = \frac{x^3 + 5x - 14}{2x^3}\end{aligned}$$

Or *simplify before differentiating...*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x - \frac{5}{2}x^{-1} + \frac{7}{2}x^{-2}\right) = \frac{1}{2} + \frac{5}{2}x^{-2} - \frac{14}{2}x^{-3}$$

Make sure that you can see that the two answers are the same.

The product rule... Explanation:

$$
\begin{aligned}
(f(x)\cdot g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x)g(x+h)+f(x)g(x+h)}^{=0}-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}\\
&\qquad\qquad\qquad + \lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x+h)+\lim_{h\to 0}f(x)\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0}g(x+h)+f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}
$$

The quotient rule... Explanation:

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x)\overbrace{-f(x)g(x) + f(x)g(x)}^{=0} - f(x)g(x+h)}{g(x)g(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)}\right)$$

$$- \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)}\right)$$

$$
\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)g(x+h)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x)g(x+h)-f(x)g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)g(x) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= \frac{1}{g(x)^2}\left(f'(x)g(x)-f(x)g'(x)\right) \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}
\end{aligned}
$$

Example. The consumption function of a small nation is given by

$$C = \frac{9Y^2 + 5Y + 100}{10Y + 1},$$

where both annual consumption C and annual income Y are measured in \$ billions.

1. Find the *marginal propensity to consume* and the *marginal propensity to save* when national income is \$8 billion.

Marginal propensity to consume: differentiate...

$$\frac{dC}{dY} = \frac{(18Y + 5)(10Y + 1) - 10(9Y^2 + 5Y + 100)}{(10Y + 1)^2}$$

$$= \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2}$$

Then evaluate:

$$\left.\frac{dC}{dY}\right|_{Y=8} = \frac{90 \cdot 64 + 18 \cdot 8 - 995}{81^2} \approx 0.7402$$

Marginal propensity to save...?

Use the '*national accounting identity*':

$$C + S = Y \quad \Longrightarrow \quad \frac{dC}{dY} + \frac{dS}{dY} = 1 \quad \Longrightarrow \quad \frac{dS}{dY} = 1 - \frac{dC}{dY}$$

So:

$$\left.\frac{dS}{dY}\right|_{Y=8} = 1 - \left.\frac{dC}{dY}\right|_{Y=8} \approx 1 - 0.7402 = 0.2598$$

2. What happens to the MPC as income continues to grow? What does this say about national consumption when income is large?

$$\lim_{Y\to\infty} \frac{dC}{dY} = \lim_{Y\to\infty} \frac{90Y^2 + 18Y - 995}{(10Y+1)^2} = \lim_{Y\to\infty} \frac{90Y^2 + 18Y - 995}{100Y^2 + 20Y + 1} = 0.9$$

Interpretation: *When income is large, the nation will tend to consume about* \$0.90 *of each additional dollar of income.*