Example 5: The revenue function for a firm's product is

$$r = 20q - 0.4q^2$$

and the firm's production function is

$$q = 5\sqrt{3\ell - 14}.$$

(*) Monthly revenue, r, is measured in \$1000s.

(*) Monthly output, q, is measured in 1000s of units.

(*) Labor input, ℓ, is measured in \$1000s per week

(*) Current labor input: $\ell_0 = 10$.

Firm is considering the hiring of a widget polisher who will cost (wages, benefits and taxes) \$500 a week. How will this affect their bottom line?

(*) Current output and revenue: $q_0 = 5\sqrt{30 - 14} = 20$ (20000 units)

and $r_0 = 20 \cdot 20 - 0.4 \cdot 20^2 = 240$ (\$240000)

(i) An increase of \$500/week in labor cost means $\Delta\ell = 0.5$

(ii) Approximate change in output:

$$\begin{aligned}
\Delta q &\approx \left.\frac{dq}{d\ell}\right|_{\ell_0=10} \cdot \Delta\ell \\
&= \frac{d}{d\ell}\left.\left(5(3\ell-14)^{1/2}\right)\right|_{\ell_0=10} \cdot (0.5) \\
&= \left.\left(5 \cdot \frac{1}{2}(3\ell-14)^{-1/2} \cdot 3\right)\right|_{\ell_0=10} \cdot (0.5) = \frac{15}{16} \quad (= 0.9375).
\end{aligned}$$

(iii) Approximate change in revenue:

$$\begin{aligned}
\Delta r &\approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \overbrace{\Delta q}^{\approx 15/16} \\
&\approx \frac{d}{dq}\left.\left(20q - 0.4q^2\right)\right|_{q_0=20} \cdot \frac{15}{16} \\
&= \left.(20 - 0.8q)\right|_{q_0=20} \cdot \frac{15}{16} = \frac{15}{4} = 3.75
\end{aligned}$$

Conclusion: Monthly revenue will increase by about \$3750 while monthly costs will increase by $\$2000 = \$500 \times 4$.

(*) *Bottom line:* Firm's profit will increase by about \$1750.

Where is the chain rule?

$$\Delta r \approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \Delta q \approx \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \overbrace{\left.\frac{dq}{d\ell}\right|_{\ell_0=10} \cdot \Delta\ell}^{\approx \Delta q} = \overbrace{\left.\frac{dr}{dq}\right|_{q_0=20} \cdot \left.\frac{dq}{d\ell}\right|_{\ell_0=10}}^{\left.\frac{dr}{d\ell}\right|_{\ell_0=10}} \cdot \Delta\ell$$

Chain rule:

$$\left.\frac{dr}{d\ell}\right|_{\ell_0=10} = \left.\frac{dr}{dq}\right|_{q_0=20} \cdot \left.\frac{dq}{d\ell}\right|_{\ell_0=10}$$

Differentiating logarithm functions.

To differentiate $y = \ln x$, we return to the definition of the derivative...

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

and simplify, using algebraic properties of $\ln x$...

$$= \lim_{h\to 0} \frac{1}{h}\left(\ln(x+h) - \ln x\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \ln\left[\left(1 + \frac{h}{x}\right)^{1/h}\right]$$

and use the *continuity* of $\ln x$...

$$= \ln\left[\lim_{h\to 0}\left(1 + \frac{h}{x}\right)^{1/h}\right]$$

What next?

Remember the special limit...

$$\lim_{u\to 0}(1+u)^{1/u} = e\ldots$$

do a little renaming...

$$\frac{h}{x} = u \implies h = ux \implies \frac{1}{h} = \frac{1}{ux} = \frac{1}{u}\cdot\frac{1}{x}\ldots$$

observe that $h \to 0$ implies $u \to 0$...

$$\begin{aligned}
\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h} &= \lim_{u\to 0}\,(1+u)^{\frac{1}{u}\cdot\frac{1}{x}} \\
&= \lim_{u\to 0}\left((1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}} \\
&= \left(\lim_{u\to 0}\,(1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}} \\
&\quad \text{(because } f(x) = a^{1/x} \text{ is continuous)} \\
&= e^{1/x}
\end{aligned}$$

Returning to $\dfrac{d}{dx}(\ln x)$...

$$\begin{aligned}
\frac{d}{dx}(\ln x) &= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} \\
&\ \ \vdots \\
&= \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right] \\
&= \ln(e^{1/x}) \\
&= \frac{1}{x}\ .
\end{aligned}$$

$$\Longrightarrow \quad \boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \quad \Longleftarrow$$

Example 1.

$$\frac{d}{dx}\left(3x^2 \ln x\right) = \overbrace{6x \ln x + 3x^2 \cdot \frac{1}{x}}^{\text{product rule}} = 6x \ln x + 3x$$

Example 2.

$$\frac{d}{dx} \ln\left(5x^2 + 3x + 1\right) = \overbrace{\frac{1}{5x^2 + 3x + 1} \cdot (10x + 3)}^{\text{chain rule}} = \frac{10x + 3}{5x^2 + 3x + 1}$$

Example 3. Differentiate $y = \ln\left(5x^2\right)$.

We can use the chain rule again: $y' = \dfrac{1}{5x^2} \cdot 10x = \dfrac{10x}{5x^2} = \dfrac{2}{x}$.

Or we can simplify and then differentiate:

$$y = \ln\left(5x^2\right) = \ln 5 + \ln x^2 = \ln 5 + 2 \ln x$$

$$\Longrightarrow \quad y' = 0 + 2 \cdot \frac{1}{x} = \frac{2}{x}$$

Observation: If $b > 0$ and $b \neq 1$, then $\log_b x = \dfrac{\ln x}{\ln b}$, so

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b} \cdot \frac{1}{x}.$$

Example 4. Find the equation of the tangent line to the graph

$$y = \log_2\left(\frac{x^2 + 2x - 1}{4x - 3}\right)$$

at the point where $x = 1$.

(*) The line passes through the point $(1, y(1)) = (1, \log_2(2)) = (1, 1)$.

(*) The slope is $y'(1)$, and again we simplify before we differentiate:

$$y = \log_2\left(\frac{x^2 + 2x - 1}{4x - 3}\right) = \log_2(x^2 + 2x - 1) - \log_2(4x - 3)$$

$$\implies y' = \frac{1}{\ln 2}\left(\frac{2x + 2}{x^2 + 2x - 1} - \frac{4}{4x - 3}\right) \implies y'(1) = -\frac{2}{\ln 2}.$$

(*) The equation of the tangent line is

$$y = 1 - \frac{2}{\ln 2}(x - 1).$$

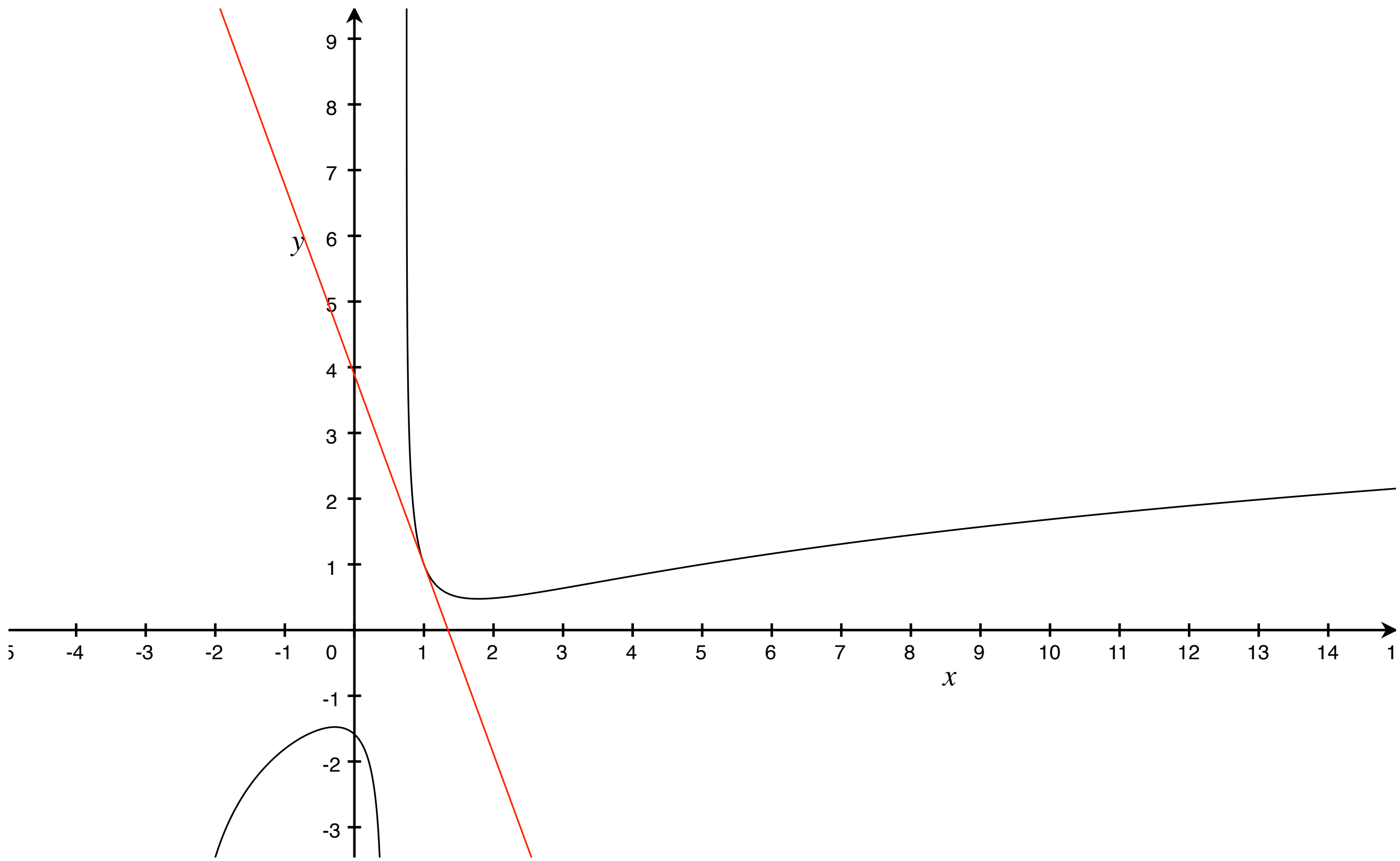

Figure 1: Graph of $y = \log_2\left(\frac{x^2+2x-1}{4x-3}\right)$ and its tangent line at $(1, 1)$

Next up (on Monday):

Logarithmic differentiation:

The chain rule tells us that

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

This is called the *logarithmic derivative* of $f(x)$.