A special limit

$$\lim_{u\to 0}(1+u)^{1/u} = e \approx 2.7182818283459.$$

We can evaluate the limit in Example 2 from Wednesday, using this limit and the following steps.

(1) $(1+0.02x)^{1/x} = (1+0.02x)^{\frac{1}{0.02x}\cdot(0.02)} = \left[(1+0.02x)^{1/(0.02x)}\right]^{0.02}$

(2) Therefore $\lim_{x\to 0}(1+0.02x)^{1/x} = \lim_{x\to 0}\left[(1+0.02x)^{1/(0.02x)}\right]^{0.02}$

$$= \left[\lim_{x\to 0}(1+0.02x)^{1/(0.02x)}\right]^{0.02}$$

because $\lim_{x\to a} f(x)^t = (\lim_{x\to a} f(x))^t$.

(3) To finish, rename $0.02x = u$ and observe that if $x \to 0$, then $u \to 0$ and vice versa, so

$$\left[\lim_{x\to 0}(1+0.02x)^{1/(0.02x)}\right]^{0.02} = \left[\lim_{u\to 0}(1+u)^{1/u}\right]^{0.02} = e^{0.02} \approx 1.02020134.$$

One-sided limits

In some cases, we want to consider the behavior of a function $f(x)$ on either side of the limiting point ***separately***.

Example: Suppose that

$$f(x) = \begin{cases} \sqrt{x} & : \quad x \geq 0 \\ \sqrt{x^2+1} & : \quad x < 0 \end{cases}$$

What can we say about $\lim_{x \to 0} f(x)$?

Since the function is defined differently on either side of $x = 0$, we have to consider the behavior on each of the two sides separately.

(*) If $x > 0$ and $x \to 0$, then $f(x) = \sqrt{x} \to 0$. We say in this case that *the limit of* $f(x)$ *as* x *approaches* 0 ***from the right*** *is equal to* 0, and write

$$\lim_{x \to 0^+} f(x) = 0$$

This is called a ***right-hand limit***.

(*) If $x < 0$ and $x \to 0$, then $f(x) = \sqrt{x^2 + 1} \to \sqrt{1} = 1$. We say in this case that *the limit of* $f(x)$ *as* x *approaches* 0 ***from the left*** *is equal to* 1, and write

$$\lim_{x \to 0^-} f(x) = 1$$

This is called a ***left-hand limit***.

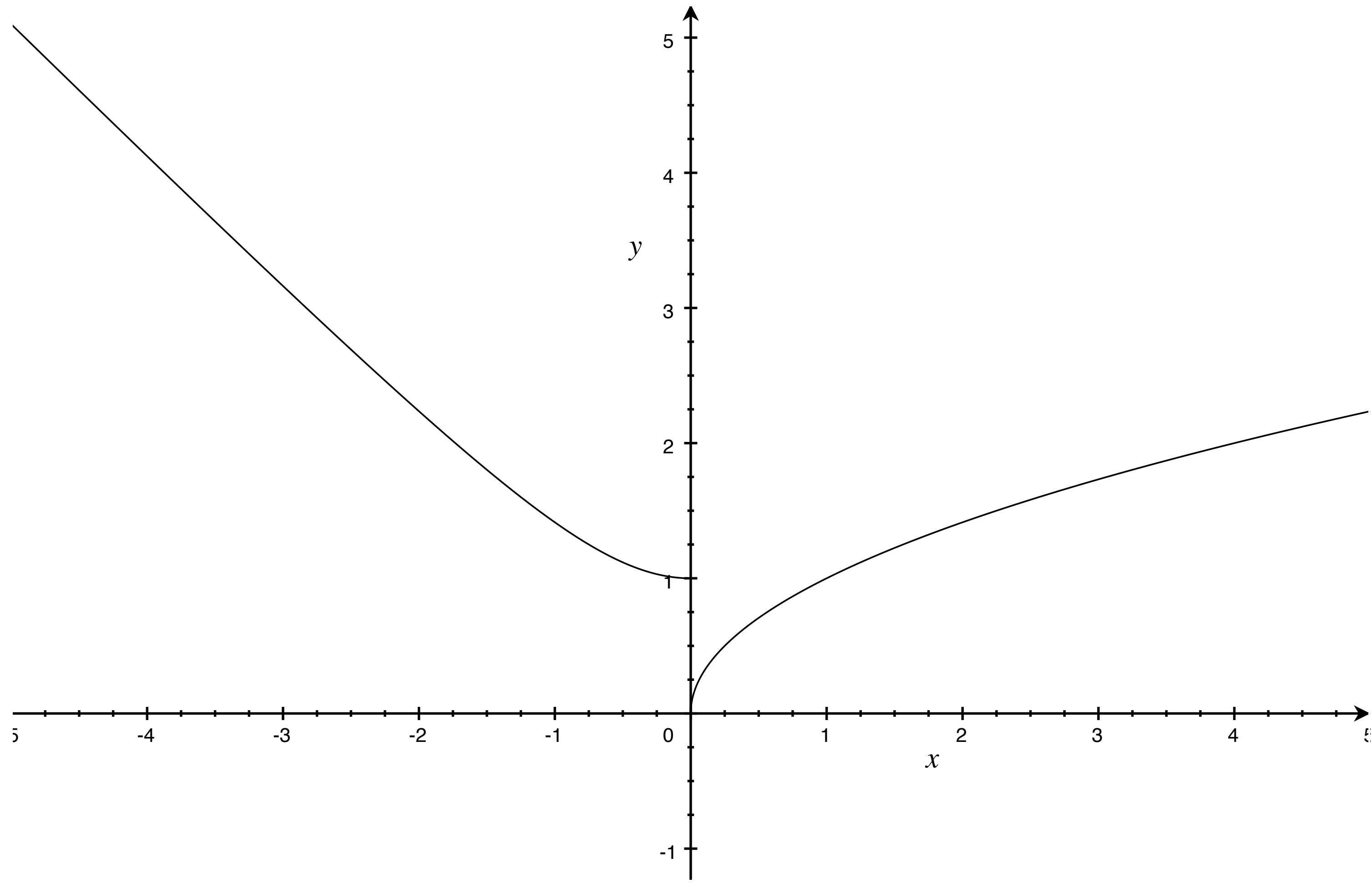
(*) In this example both the left- and right-hand limits exist, but they are *different from each other*,

$$\lim_{x \to 0^-} f(x) = 1 \neq 0 = \lim_{x \to 0^+} f(x),$$

so the two-sided limit

$$\lim_{x \to 0} f(x) \quad \textit{does not exist},$$

because there is no *single number* L that $f(x)$ approaches as $x \to 0$.



$$\text{Graph of } y = \begin{cases} \sqrt{x} & : \quad x \geq 0 \\ \sqrt{x^2+1} & : \quad x < 0 \end{cases}$$

Definitions:

The limit of $f(x)$ as x approaches a ***from the right*** is equal to L, written

$$\lim_{x \to a^+} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a *from the right*, i.e., $x > a$.

The limit of $f(x)$ as x approaches a ***from the left*** is equal to L, written

$$\lim_{x \to a^-} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a *from the left*, i.e., $x < a$.

Observation:

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).$$

Example: Suppose that

$$g(x) = \begin{cases} 2 + 2x - x^2 & : \quad x \geq 1 \\ \dfrac{x^2 + x - 2}{x - 1} & : \quad x < 1 \end{cases}$$

Find the limit $\lim_{x \to 1} g(x)$, or explain why it does not exist.

(*) $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 2 + 2x - x^2 = 2 + 2 - 1 = 3.$

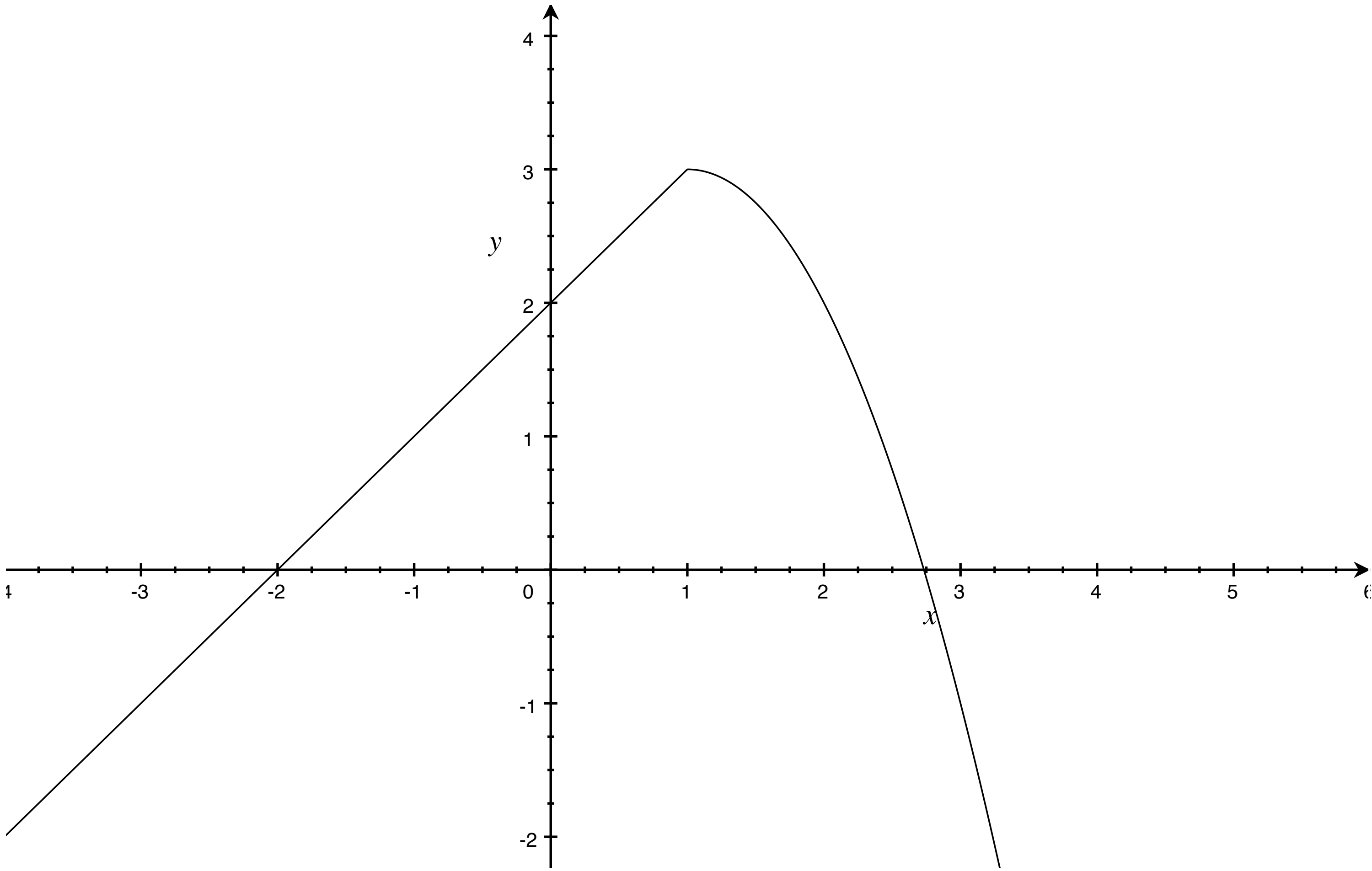
and

(*) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \dfrac{x^2 + x - 2}{x - 1} = \lim_{x \to 1^-} \dfrac{\cancel{(x-1)}(x+2)}{\cancel{x-1}}$

$= \lim_{x \to 1^-} x + 2 = 1 + 2 = 3$

Therefore

$$\lim_{x \to 1} g(x) = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) = 3$$



Graph of $y = \begin{cases} 2 + 2x - x^2 & : \quad x \geq 1 \\ \dfrac{x^2 + x - 2}{x - 1} & : \quad x < 1 \end{cases}$

‘Infinite’ limits.

If $f(x)$ grows larger and larger without bound as x approaches some point a, then we say that $f(x)$ *is approaching infinity* as x approaches a and write

$$\lim_{x \to a} f(x) = \infty.$$

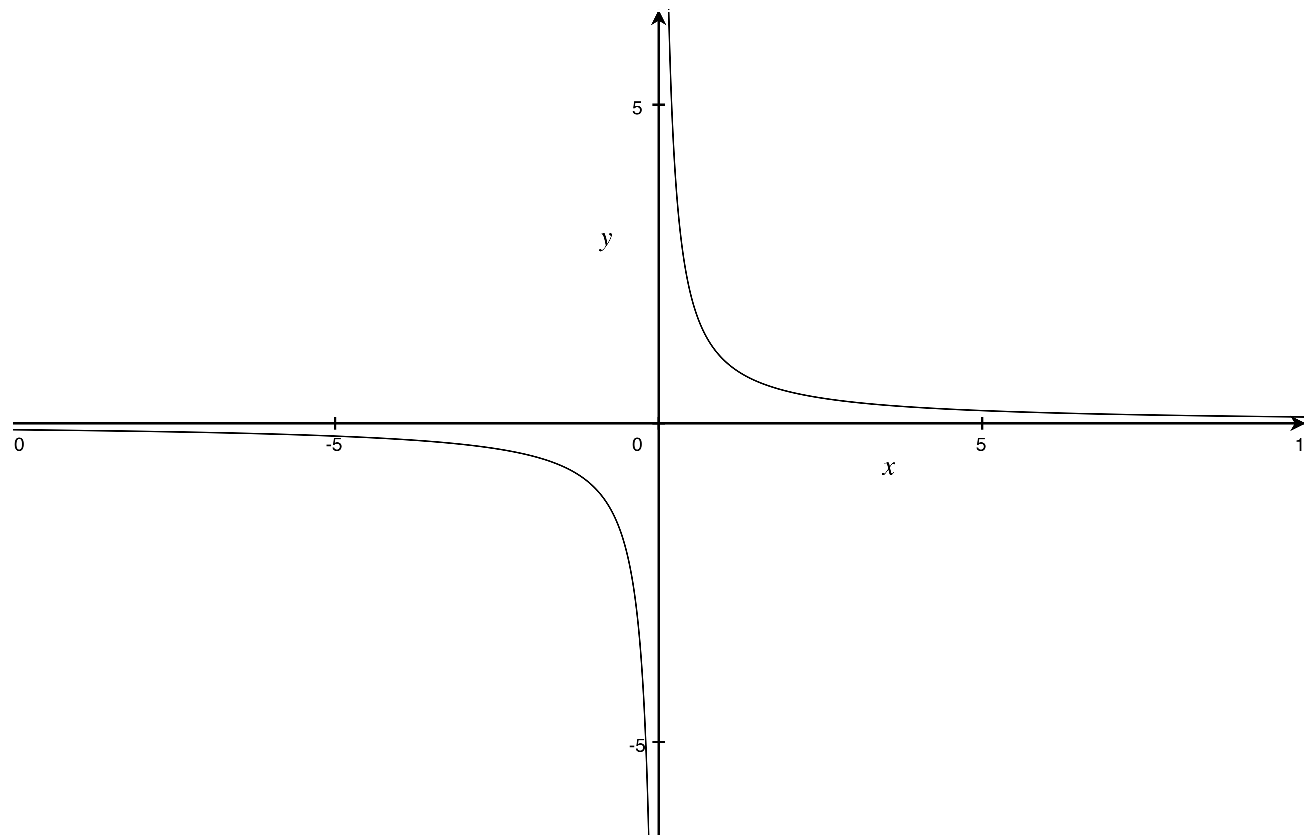
Likewise, If $f(x)$ grows more and more negative without (lower) bound as x approaches some point a, then we say that $f(x)$ *is approaching negative infinity* as x approaches a and write

$$\lim_{x \to a} f(x) = -\infty.$$

(*) If $f(x)$ exhibits one of these behaviors on only one side of a or the other, then we use one-sided limits to describe the situation.

Example: $\displaystyle\lim_{x \to 0} \frac{1}{x}$ does not exist, but we can say that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$



Comment: Even though we might say that $\lim_{x\to a} f(x) = \pm\infty$, the limit $\lim_{x\to a} f(x)$ still ***does not exist***. On the other hand, writing

$$\lim_{x\to a} f(x) = \pm\infty$$

conveys more information than simply saying that the limit does not exist.

Limits 'at infinity'

(*) The *limit at infinity* of a function $f(x)$ describes (certain aspects of) the behavior of $f(x)$ as the variable x grows larger and larger.

Example 1: What happens to the values of $f(x) = \dfrac{1}{x}$ as x grows without bound?

x	$\frac{1}{x}$
1	1
100	0.01
$10,000$	0.0001
$1,000,000$	0.000001
10^{100}	$0.\overbrace{00\ldots0}^{99\text{ 0s}}1$

Observations:

(i) As k grows bigger (and 10^k grows even faster), $\frac{1}{10^k}$ approaches 0.

(ii) If $10^k < x$, then $0 < \frac{1}{x} < \frac{1}{10^k}$, so $\frac{1}{x}$ approaches 0 as x grows large.

Definition:

The limit of $f(x)$ as x 'approaches infinity' is equal to L, written

$$\lim_{x\to\infty} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x grows larger and larger.

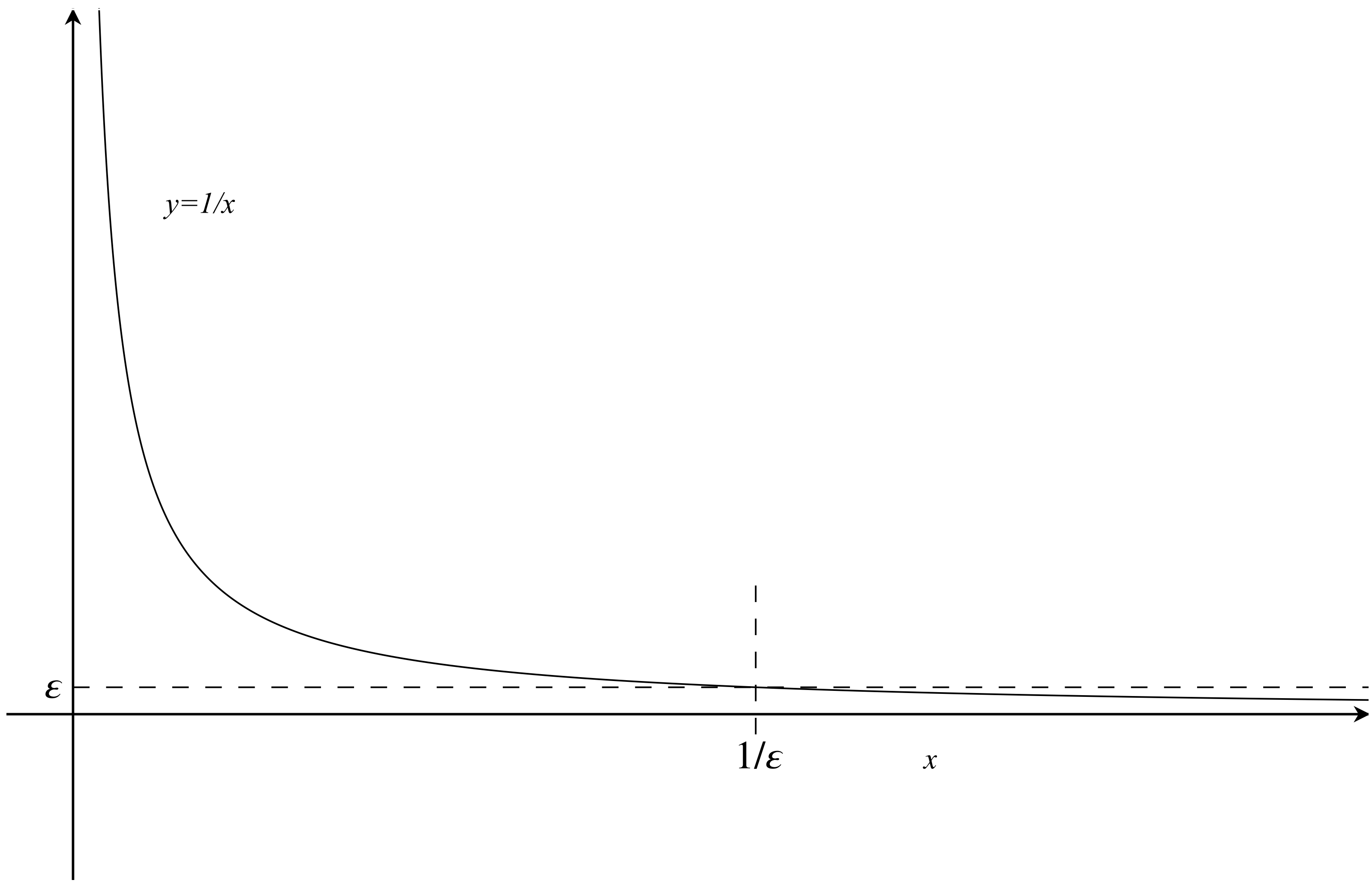
In other words, 'approaching infinity' means getting large without bound.

More formal definition:

$\lim_{x\to\infty} f(x) = L$ means that given $\varepsilon > 0$, there is an M such that $|f(x) - L| < \varepsilon$ once $x > M$.

Example: $\lim_{x\to\infty} \frac{1}{x} = 0$ because given $\varepsilon > 0$, if $x > M = \frac{1}{\varepsilon}$, then

$$\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \frac{1}{1/\varepsilon} = \varepsilon.$$


y=1/x
ε
1/ε
x

1. The rules for constant functions and for sums and differences of limits are valid for limits at infinity.

2. The rules for products and quotients of limits are equally valid for limits at infinity, *as long as the component limits are finite.*

(*) **Comment:** The expressions $\dfrac{\infty}{\infty}$ and $0 \cdot \infty$ are as meaningless as $\dfrac{0}{0}$.

3. If $k > 0$, then $\lim\limits_{x \to \infty} x^k = \infty$, i.e., x^k grows larger as x grows larger.

4. If $k < 0$, then $\lim\limits_{x \to \infty} x^k = \lim\limits_{x \to \infty} \dfrac{1}{x^{|k|}} = 0$.

5. More generally, if $\lim\limits_{x \to \infty} f(x) = \infty$, then $\lim\limits_{x \to \infty} \dfrac{1}{f(x)} = 0$.

6. If $a > 0$, then $\lim\limits_{x \to \infty} e^{ax} = \infty$ and therefore $\lim\limits_{x \to \infty} e^{-ax} = \lim\limits_{x \to \infty} \dfrac{1}{e^{ax}} = 0$.

7. *Exponential growth.* If $a > 0$, then for any k

$$\lim_{x \to \infty} \frac{x^k}{e^{ax}} = 0.$$

Limits at infinity of *rational functions*.

Example. Find $\lim_{x\to\infty} \frac{2x^2+10x+100}{x^3+x^2+1}$.

(*) The rule for quotients doesn't work here, because

$$\lim_{x\to\infty} 2x^2+10x+100 = \infty = \lim_{x\to\infty} x^3+x^2+1.$$

However, the denominator is *growing faster* than the numerator (why?), which indicates that the limit is probably 0.

To see that this is true, we use a little algebra to simplify:

$$\frac{2x^2+10x+100}{x^3+x^2+1} = \frac{\cancel{x^2}\left(2+\frac{10}{x}+\frac{100}{x^2}\right)}{x^{\cancel{3}}\left(1+\frac{1}{x}+\frac{1}{x^3}\right)} = \frac{1}{x}\cdot\frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}}$$

Therefore...

$$
\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+10x+100}{x^3+x^2+1} &= \lim_{x\to\infty} \left(\frac{x^2}{x^3} \cdot \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \left(\frac{1}{x} \cdot \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \right) \\
&= \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{2+\frac{10}{x}+\frac{100}{x^2}}{1+\frac{1}{x}+\frac{1}{x^3}} \\
&= 0 \cdot \frac{\lim\limits_{x\to\infty} 2+\dfrac{10}{x}+\dfrac{100}{x^2}}{\lim\limits_{x\to\infty} 1+\dfrac{1}{x}+\dfrac{1}{x^3}} \\
&= 0 \cdot \frac{2+0+0}{1+0+0} = 0 \cdot 2 = 0.
\end{aligned}
$$

Observation: If U is growing (very) large and $a > 0$, then aU will also grow (very) large. On the other hand, if $a < 0$, then aU will grow (very) negative, i.e., $aU < 0$ and $|aU|$ is growing (very) large.

Conclusions: If $\lim_{x\to\infty} f(x) = \infty$, then

$$\lim_{x\to\infty} af(x) = \infty$$

if $a > 0$ and

$$\lim_{x\to\infty} af(x) = -\infty$$

if $a < 0$.

Moreover, if $\lim_{x\to\infty} g(x) = a > 0$, then

$$\lim_{x\to\infty} g(x)f(x) = \infty$$

and if $\lim_{x\to\infty} g(x) = b < 0$, then

$$\lim_{x\to\infty} g(x)f(x) = -\infty$$

Example: Find $\lim_{x\to\infty} \dfrac{x^4+10x-5}{2x^3+x^2+1}$.

Simplify as before:

$$\frac{x^4+10x-5}{2x^3+x^2+1} = \frac{\cancel{x^4}\left(1+\frac{10}{x^3}-\frac{5}{x^4}\right)}{\cancel{x^3}\left(2+\frac{1}{x}+\frac{1}{x^3}\right)} = x\cdot\frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}}$$

Therefore

$$\lim_{x\to\infty}\frac{x^4+10x-5}{2x^3+x^2+1} = \lim_{x\to\infty}\left(x\cdot\frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}}\right)$$

Now, $\lim_{x\to\infty} x = \infty$ and

$$\lim_{x\to\infty}\frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty} 1+\frac{10}{x^3}-\frac{5}{x^4}}{\lim_{x\to\infty} 2+\frac{1}{x}+\frac{1}{x^3}} = \frac{1+0-0}{2+0+0} = \frac{1}{2} > 0$$

so it follows that

$$\lim_{x\to\infty}\left(x\cdot\frac{1+\frac{10}{x^3}-\frac{5}{x^4}}{2+\frac{1}{x}+\frac{1}{x^3}}\right) = \infty.$$