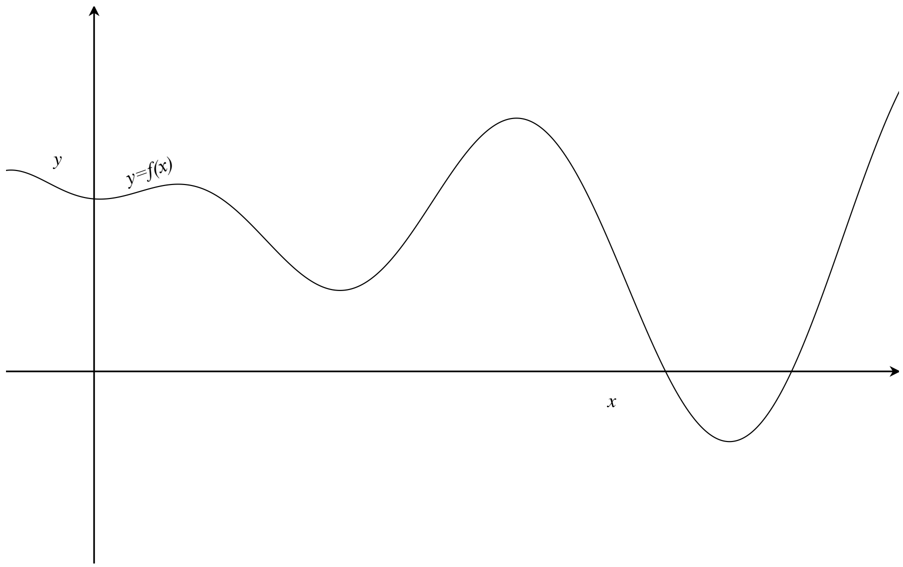
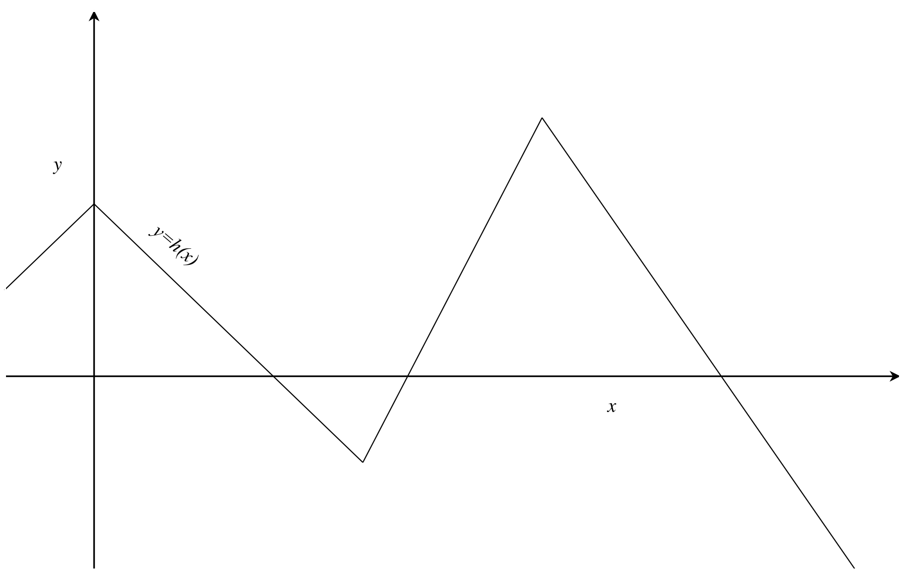
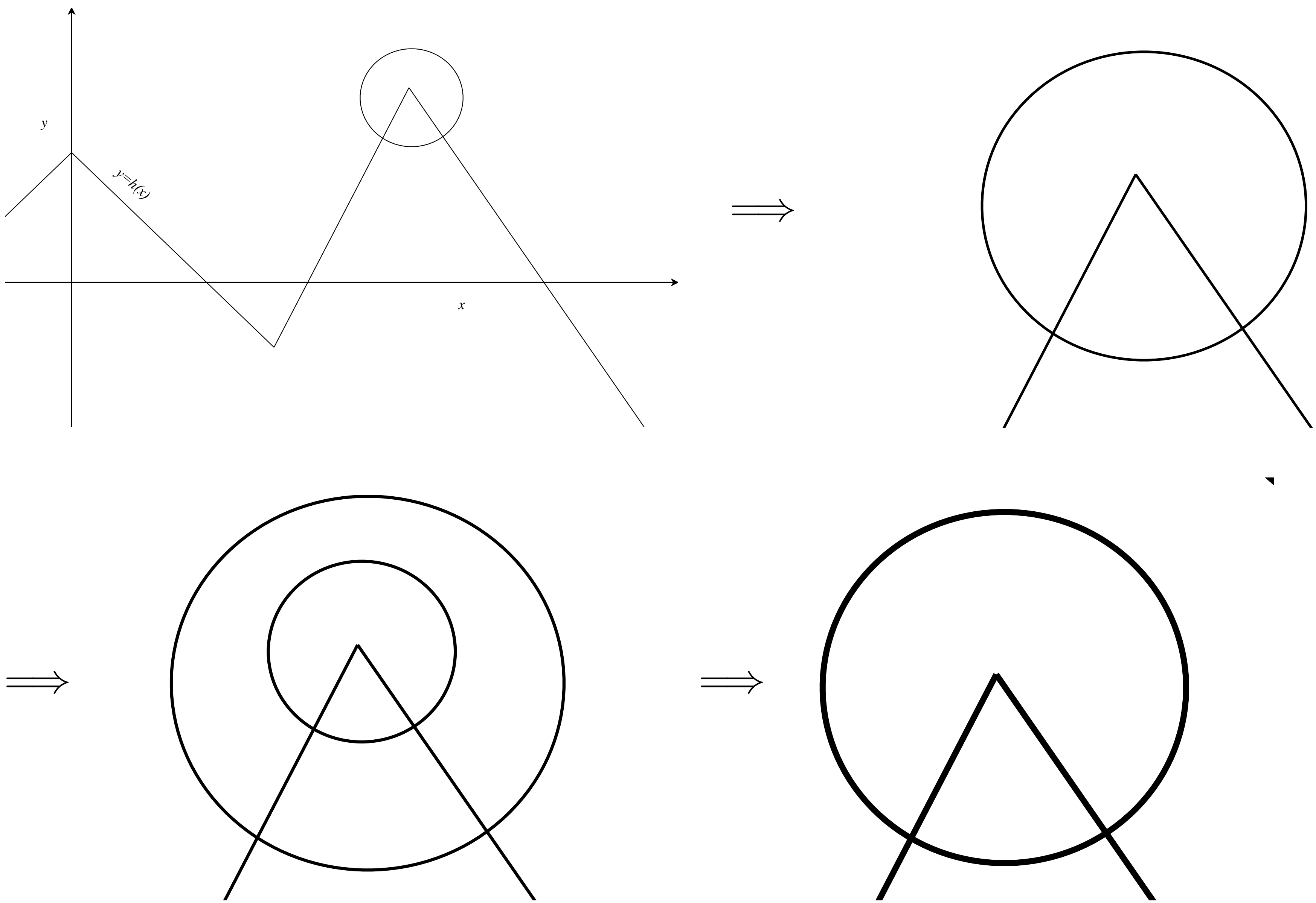
Continuous vs. Smooth

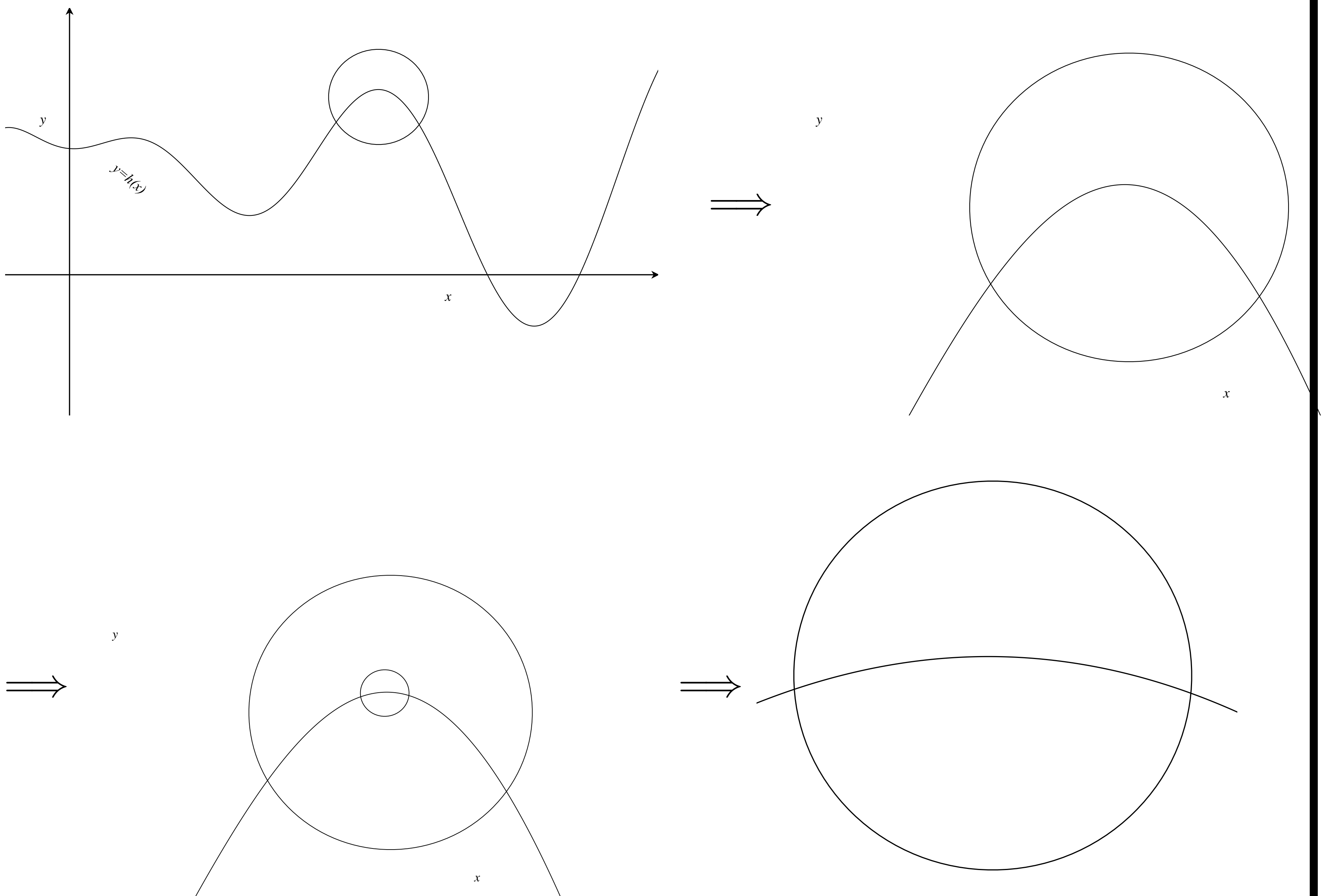




Zooming on the sawtooth graph:



Zooming in on the ‘smooth’ curve...



Differences:

1. The ‘smooth graph’ changes directions gradually, while the ‘sawtooth’ graph changes directions abruptly (at some points), ‘without warning’.

2. By zooming in, it seems possible that we can associate a specific direction, or slope to every point on the ‘smooth’ graph, but at the points where the ‘sawtooth’ graph changes direction, there is no well-defined direction.

3. In fact, zooming in on the ‘smooth’ graph makes the graph flatter (straighter) around each point. When we zoom in on the corners of the sawtooth graph, the corner never goes away and the graph is never flat there.

This is worth repeating: As we zoom in to a point on the smooth graph, the graph becomes more and more like a straight line there. The slope (direction) of that straight line is what we will eventually call *the slope of the graph* at that point.

Definition: *The **tangent line** to the graph* $y = f(x)$ *at the point* $(x_0, f(x_0))$*, is the line that passes through the point* $(x_0, f(x_0))$ *and has the same slope as the graph at that point.*

Question: How do we know that the graph has a slope, and if it does, how do we find it?

Recall that the slope of a *straight line* is defined by the *difference quotient*

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

where (x_0, y_0) and (x_1, y_1) are points on the line satisfying $x_1 \neq x_0$.

(*) If no such points exist on the line, then the line is vertical and technically does not have a slope.

The problem we encounter when we try to define the slope of the graph $y = f(x)$ (which is not a straight line) at the point $(x_0, f(x_0))$ is that we only have one point to work with.

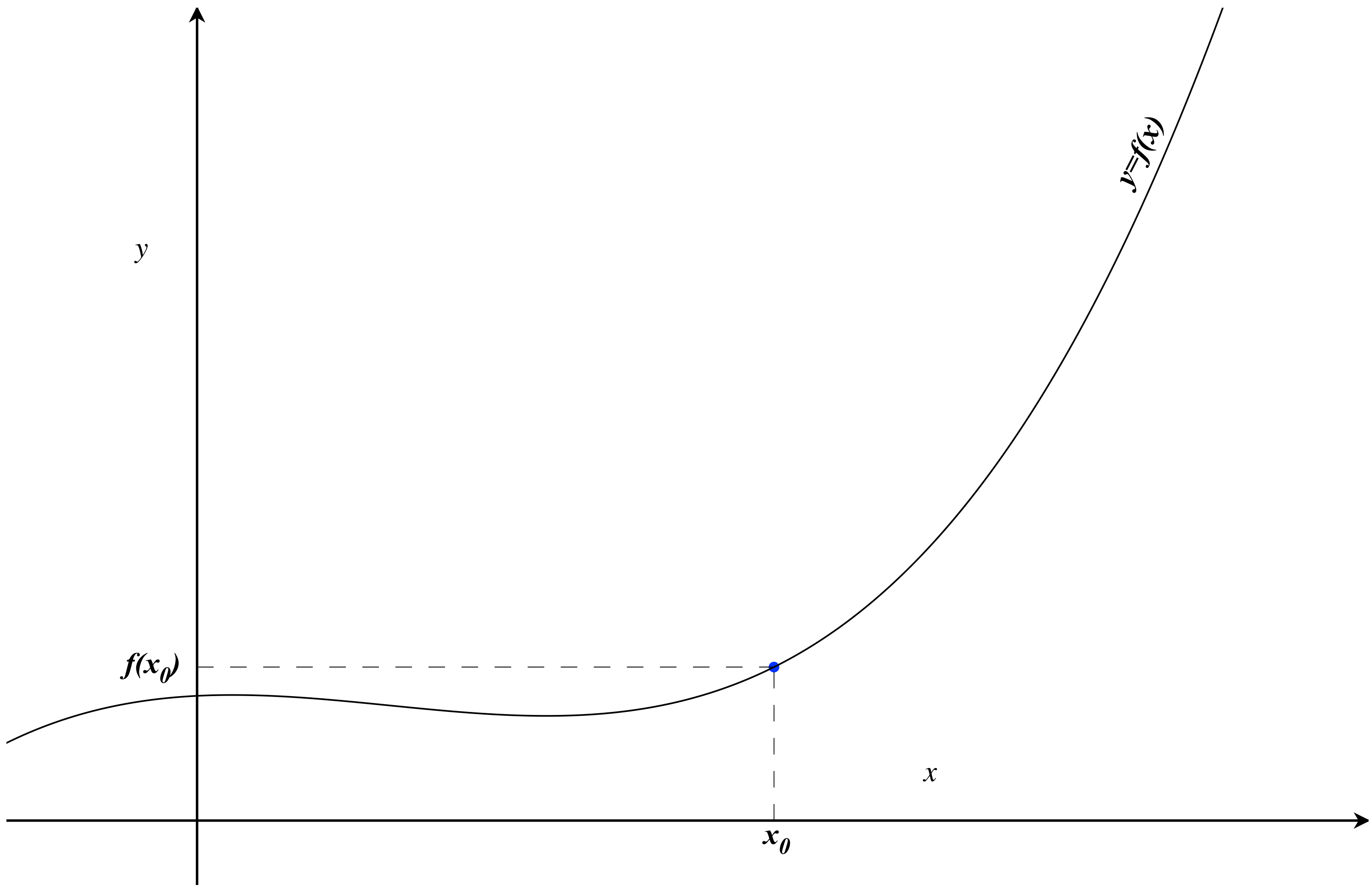
The solution to this problem, is to

(i) find an approximate value for the slope at the given point,

(ii) find a way to repeatedly improve the approximation and

(iii) compute the limit of the (improving) approximations. This limit, ***if it exists***, will be the slope we seek.

Question: How do we find the slope of the graph $y = f(x)$ at the point $(x_0, f(x_0))$ on the graph?



Step 1. *Find an approximation to the slope.*

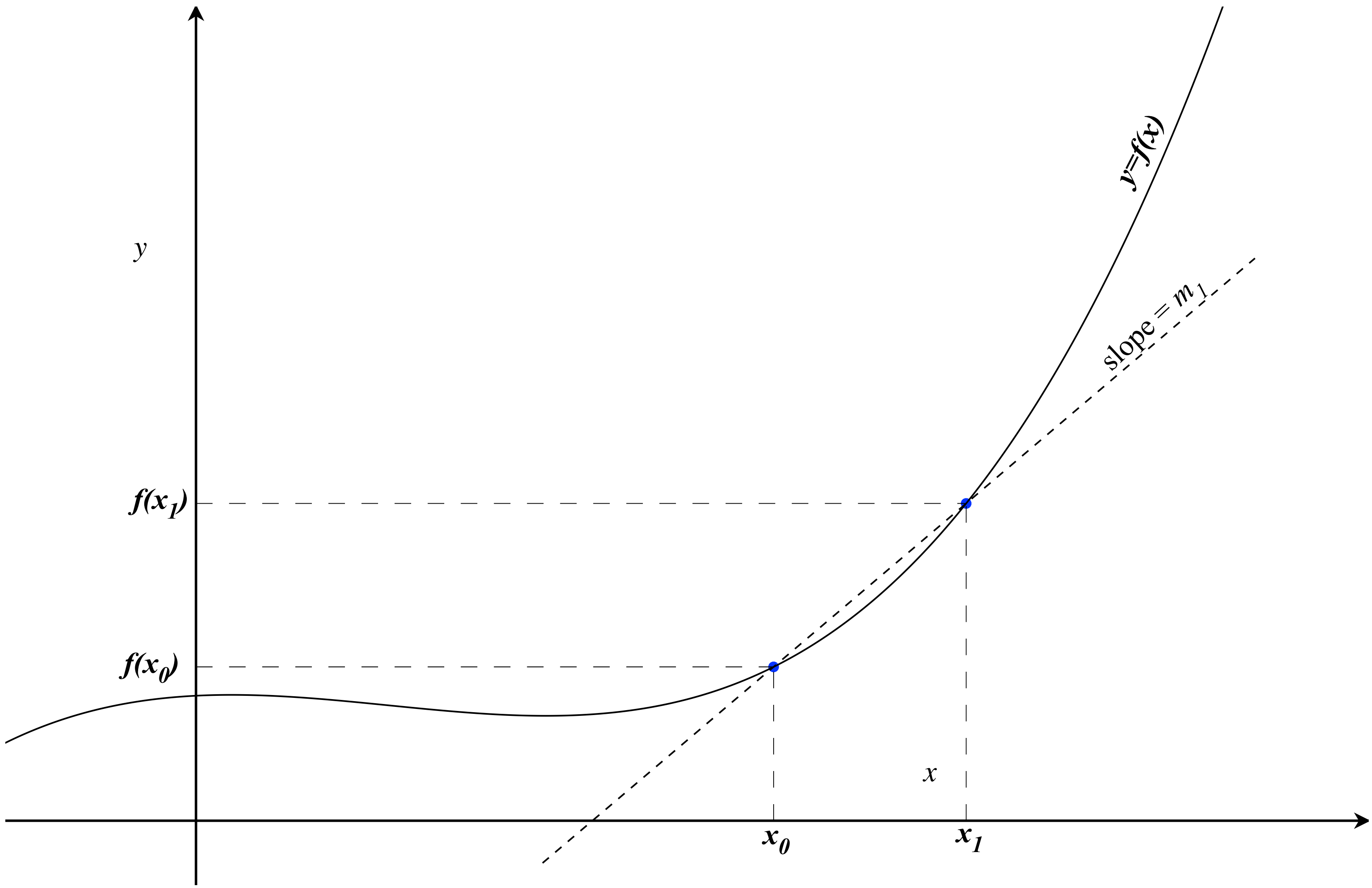
Problem: we need two points to calculate a slope but we have only one, $(x_0, f(x_0))$.

Solution: choose another point on the graph, say $(x_1, f(x_1))$, and calculate the slope of the (secant) line that connects these two points:

$$m_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The slope m_1 is an approximation of the (unknown) slope that we seek.

Step 1. (illustrated)



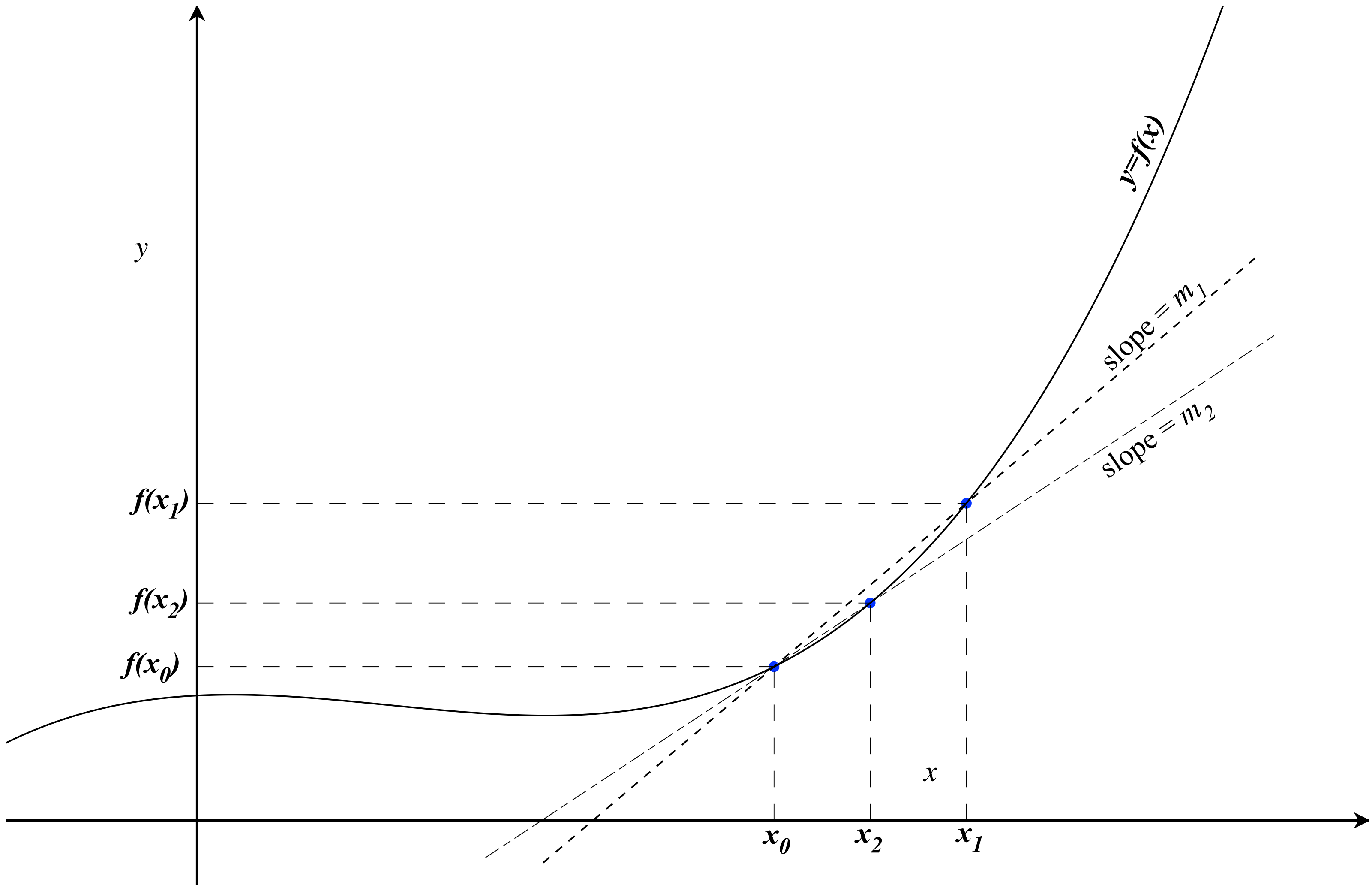
Step 2. *Find a (repeatable) way to improve the approximation.*

Intuition: If we choose a point x_2 that is ***closer*** to x_0 than x_1, and find the slope

$$m_2 = \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

of the line connecting $(x_0, f(x_0))$ to $(x_2, f(x_2))$, then m_2 will be a better approximation to the slope ***at*** $(x_0, f(x_0))$ than m_1...

Step 2. (illustrated)



Step 3 and beyond: *Repeat step 2 and take a limit.*

Intuition: If we continue to choose points on the graph that are closer and closer to $(x_0, f(x_0))$ and compute the slopes of the secant lines connecting these points to $(x_0, f(x_0))$, then these slopes should approach the slope ***at*** $(x_0, f(x_0))$, ***if it exists.***

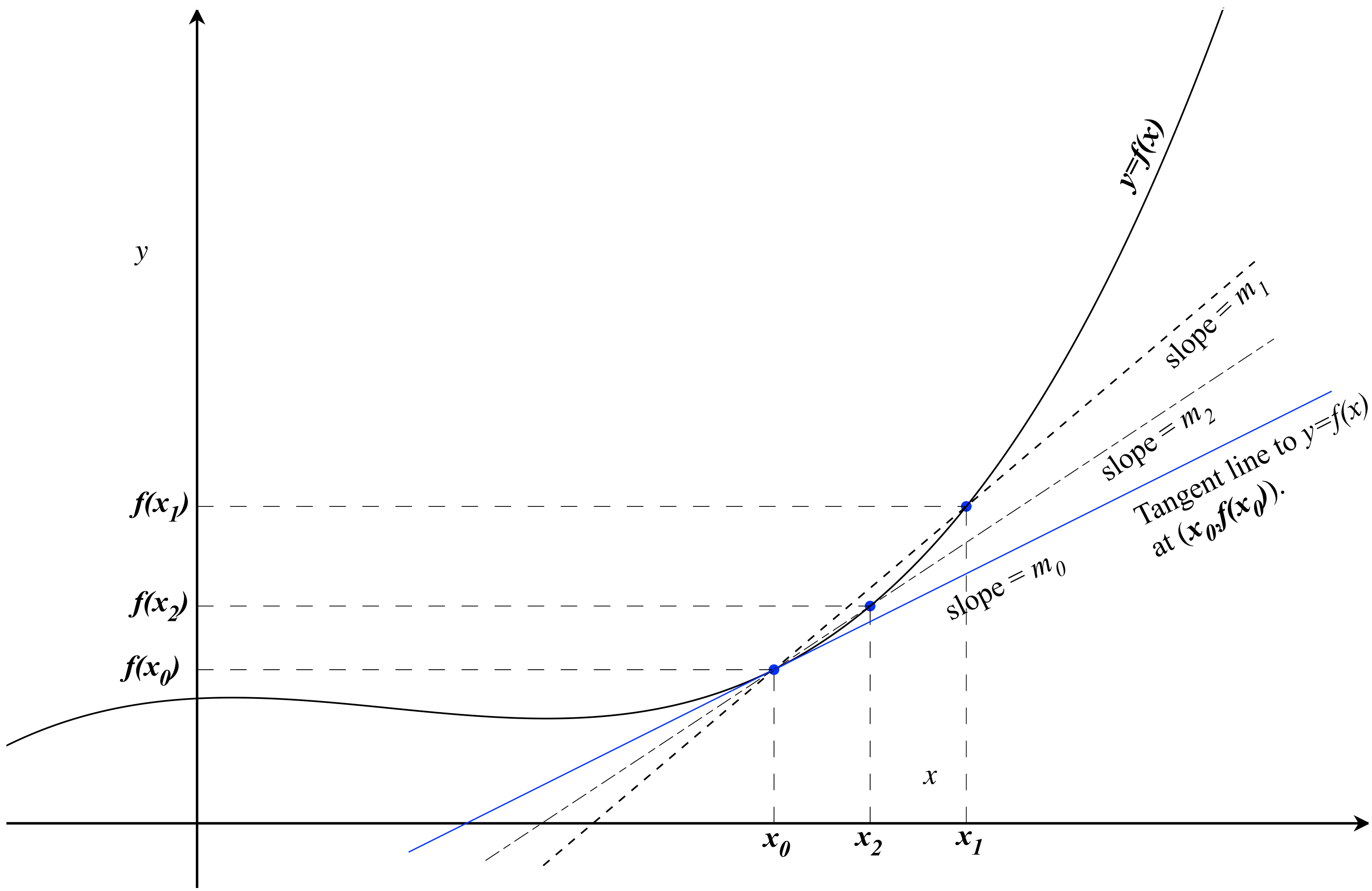
Definition: *The slope of the graph* $y = f(x_0)$ *at the point* $(x_0, f(x_0))$ *is given by*

$$m_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

if the limit exists*. If the limit does not exist, then the graph does not have a slope at that point.*

Definition: *If the graph* $y = f(x)$ *has a slope* m_0 *at* $(x_0, f(x_0))$*, then the straight line that passes through* $(x_0, f(x_0))$ *with slope* m_0 *is called the* ***tangent line*** *to the graph at that point.*

Tangent line.

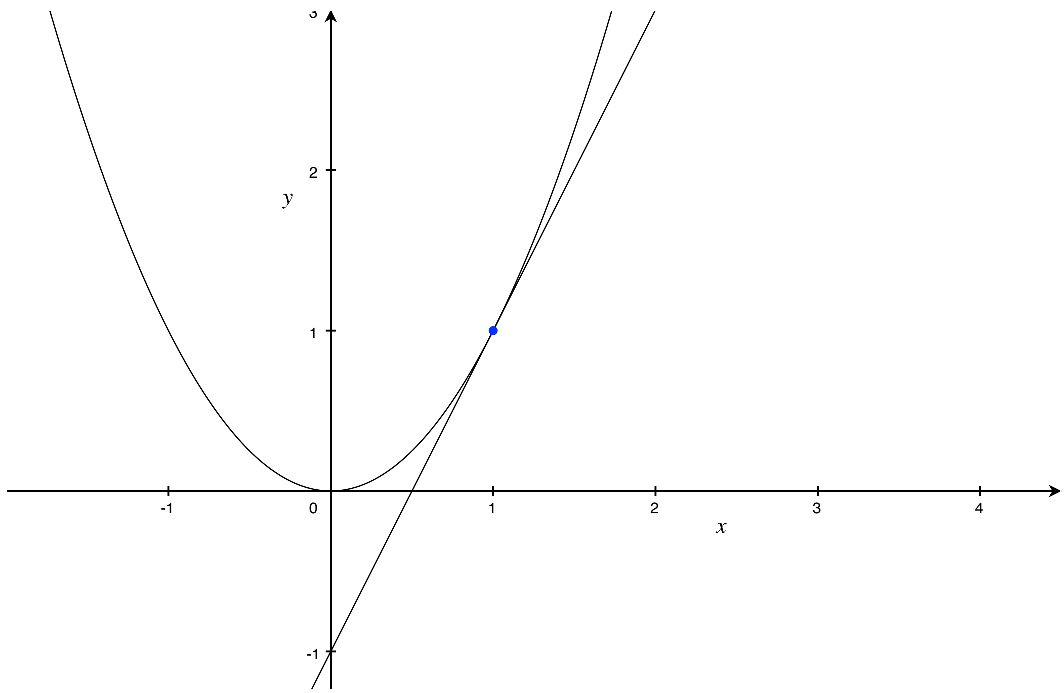


Example. Find the slope of the graph $y = x^2$ at the point $(1, 1)$, and find the equation of the tangent line to $y = x^2$ at $(1, 1)$.

$$m = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} x + 1 = 2.$$

The tangent line we seek passes through the point $(1, 1)$ with slope $m = 2$. To find its equation, we use the *point-slope* formula:

$$y - 1 = 2(x - 1) \implies y = 2x - 1.$$



Terminology and Notation:

The limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

(if it exists) is called the ***derivative*** of the function $y = f(x)$ at the point x_0, and is denoted by $f'(x_0)$ (among others). I.e.,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example. (Revisited) If $f(x) = x^2$ and $x_0 = 1$, then

$$f'(1) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1} x + 1 = 2.$$

First interpretation of the derivative: The slope of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is given by the derivative $f'(x_0)$. The equation of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0) \implies y = f(x_0) + f'(x_0)(x - x_0).$$

Modifying the way we express the limit. If we write $x = x_0 + h$, then $x - x_0 = h$, and $x \to x_0$ is the same as $h \to 0$. This means that we can rewrite the definition of the derivative as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example. (Revisited again) The derivative of $f(x) = x^2$ at $x = 1$ is

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} 2 + h = 2. \end{aligned}$$

Example. (A new one, finally) Find the derivative of $y = x^2$ at the points $x = 2$ and $x = -1$.

$$\begin{aligned} y'(2) &= \lim_{h\to 0} \frac{(2+h)^2 - 4}{h} \\ &= \lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}} = \lim_{h\to 0} 4 + h = 4. \end{aligned}$$

and

$$\begin{aligned} y'(-1) &= \lim_{h\to 0} \frac{(-1+h)^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{\cancel{1} - 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-2+h)}{\cancel{h}} = \lim_{h\to 0} -2 + h = -2. \end{aligned}$$

Definition: The derivative (function) of $y = f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

at every point x where the limit exists.

Terminology: If $f'(x)$ is defined at a point x_0, then the function $f(x)$ is said to be *differentiable* at x_0. If $f'(x)$ exists for every point x in some interval $I = (a, b)$, then $f(x)$ is differentiable in I.

Example. Find the derivative (function) of the function $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x. \end{aligned}$$

This limit exists for all x, so $f(x) = x^2$ is differentiable for all x.