Vertical asymptotes.

Definition: If the function $y = f(x)$ is not defined at $x = a$ ***and***

$$\lim_{x \to a^+} = \pm\infty \quad \text{and} \quad \lim_{x \to a^-} = \pm\infty,$$

then the line $x = a$ is a ***vertical asymptote*** to the graph $y = f(x)$.

Horizontal asymptotes.

Definition: If $\lim_{x \to \infty} f(x) = \alpha$ or $\lim_{x \to -\infty} f(x) = \alpha$, then the line $y = \alpha$ is a ***horizontal asymptote*** to the graph $y = f(x)$.

Oblique asymptotes.

Definition:

The line $y = ax + b$ is an ***oblique asymptote*** to the graph $y = f(x)$ if

- $\lim_{x\to\infty} [f(x) - (ax + b)] = 0.$
  I.e., the two graphs approach each other as x tends to infinity... and
- $\lim_{x\to\infty} f'(x) = a.$
  I.e., the slopes of both graphs approach each other as x tends to infinity.

To find the oblique asymptote to the graph of $y = f(x)$ ...

1. If $\lim_{x\to\infty} f'(x)$ doesn't exist, then there is no oblique asymptote.
2. If $\lim_{x\to\infty} f'(x) = a$ (exists), then there is an oblique asymptote. In this case...
3. Find $\lim_{x\to\infty}(f(x) - ax) = b$... The line $y = ax + b$ is the desired oblique asymptote.

Example: Sketch the graph of the function $f(x) = \dfrac{x^2+2x+5}{2x-1}$.

Points of interest. (Intercepts, critical points, inflection points, etc.)

Intercepts: y-intercept: $(0,-5)$. No x-intercepts (why?).

Interesting point: The function is not defined at $x = 1/2$.

Critical points:

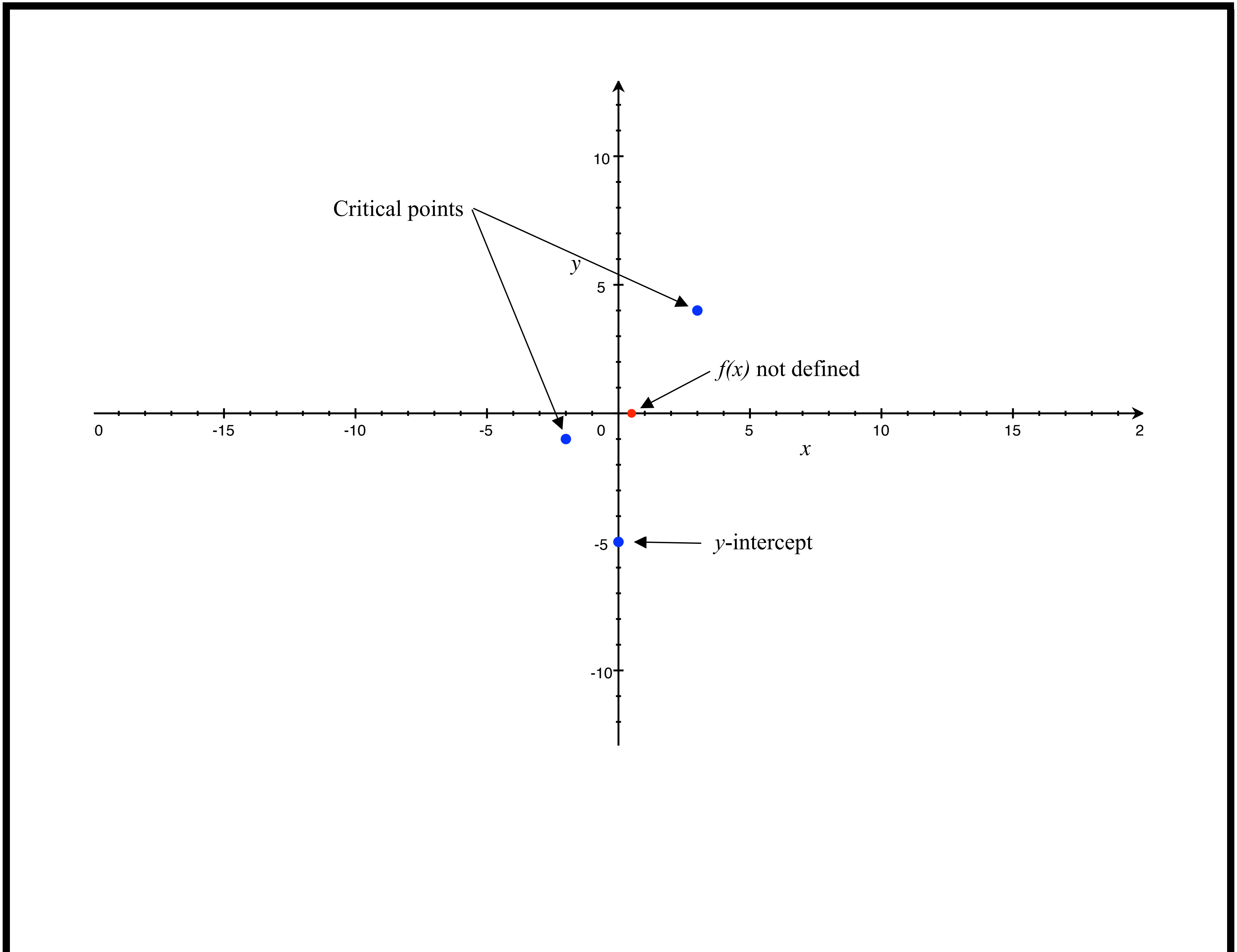
$$f'(x) = \frac{(2x+2)(2x-1) - 2(x^2+2x+5)}{(2x-1)^2} = \frac{2x^2-2x-12}{(2x-1)^2}$$

$f'(x) = 0 \implies 2x^2 - 2x - 12 = 0 \implies x_1 = -2$ and $x_2 = 3$.

Points: $(-2, f(-2)) = (-2,-1)$ and $(3, f(3)) = (3,4)$.

Inflection points: None, because:

$$\begin{aligned} f''(x) &= \frac{(4x-2)(2x-1)^2 - 4(2x-1)(2x^2-2x-12)}{(2x-1)^4} \\ &= \frac{(4x-2)(2x-1) - 4(2x^2-2x-12)}{(2x-1)^3} = \frac{50}{(2x-1)^3} \end{aligned}$$

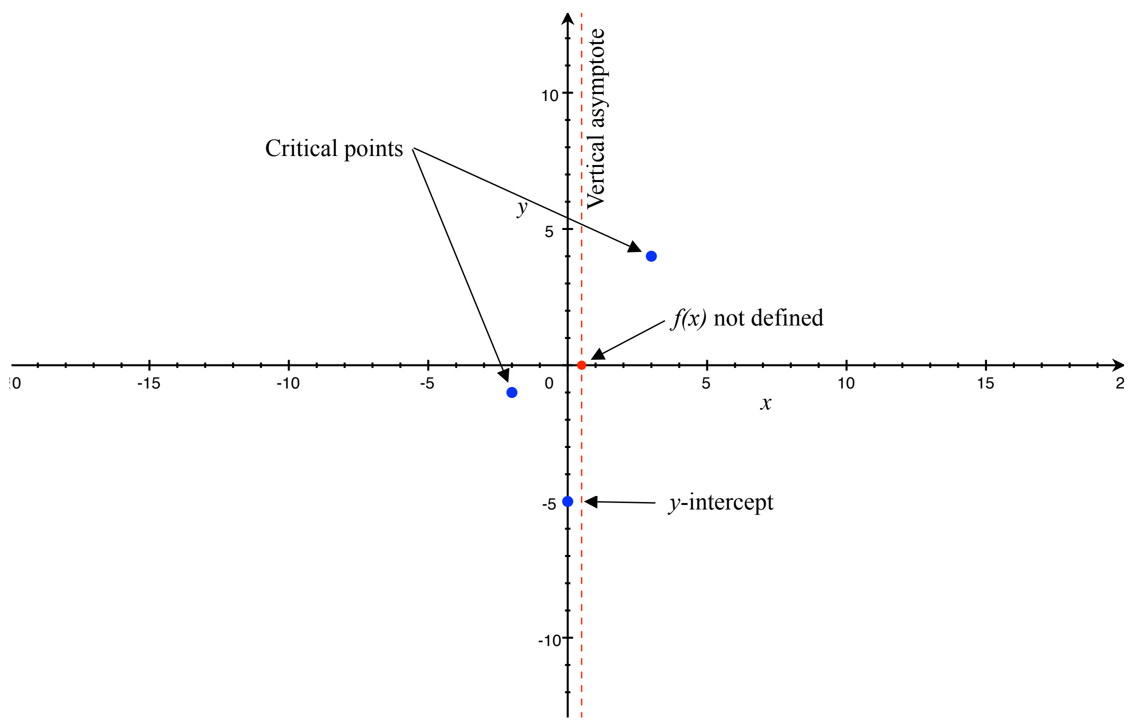

Critical points
f(x) not defined
y-intercept
y
x
10
5
-5
-10
0
-15
-10
-5
0
5
10
15
2

Vertical asymptote. We have a point where the function is not defined, so we evaluate the one-sided limits on either side of this point:

$$\lim_{x \to \frac{1}{2}^+} \frac{x^2 + 2x + 5}{2x - 1} = \infty \text{ (because } f(x) > 0 \text{ if } x > \tfrac{1}{2}\text{).}$$

$$\lim_{x \to \frac{1}{2}^-} \frac{x^2 + 2x + 5}{2x - 1} = -\infty \text{ (because } f(x) < 0 \text{ if } x < \tfrac{1}{2}\text{).}$$

Conclusion: the line $x = 1/2$ is a *vertical asymptote* to the graph $y = f(x)$.



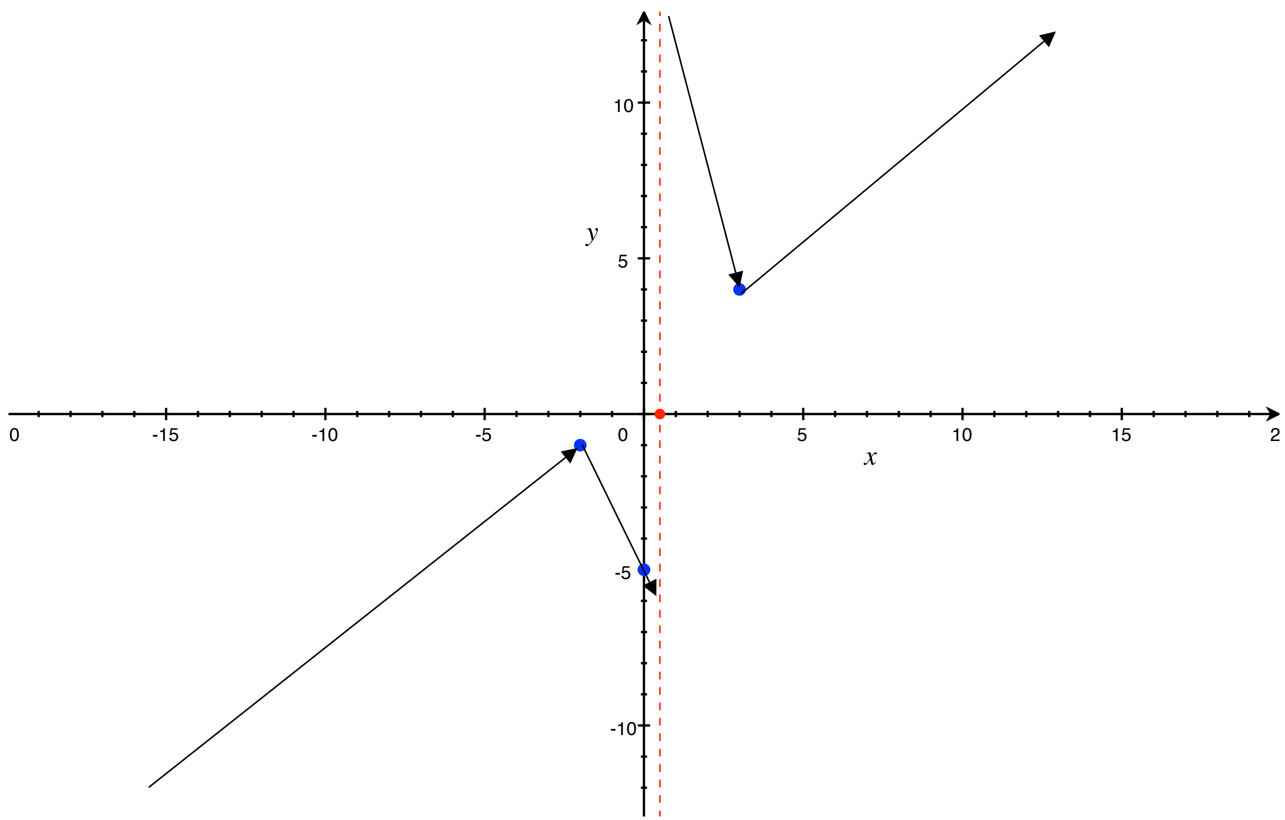
Increasing and decreasing.

(*) $f'(-3) > 0$, so $f(x)$ is increasing on $(-\infty, -2)$;

(*) $f'(0) < 0$, so $f(x)$ is decreasing on $(-2, 1/2)$;

(*) $f'(1) < 0$, so $f(x)$ is decreasing on $(1/2, 3)$;

(*) $f'(4) > 0$, so $f(x)$ is increasing on $(3, \infty)$;



Concavity. $f''(x) = \dfrac{50}{(2x-1)^3}$ so the graph $y = f(x)$ is concave up in $(1/2, \infty)$ and concave down in $(-\infty, 1/2)$.

Oblique asymptote.

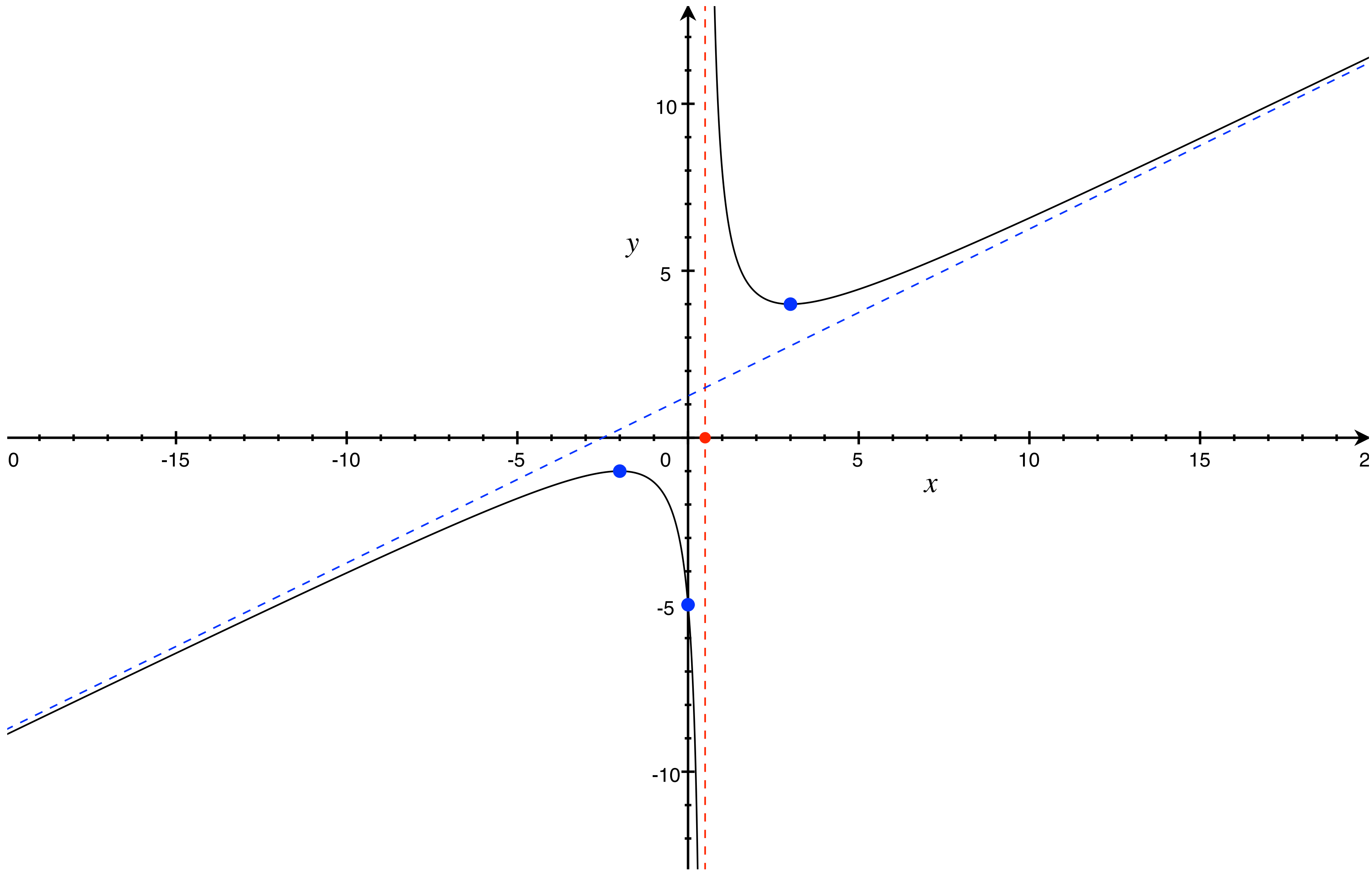
First: $\displaystyle \lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{2x^2-2x-12}{(2x-1)^2} = \lim_{x\to\infty} \frac{2x^2-2x-12}{4x^2-4x+1} = \frac{1}{2}$

so there is an oblique asymptote with slope $1/2$.

$$\text{Second:} \quad \lim_{x\to\infty}\left(f(x) - \frac{1}{2}x\right) = \lim_{x\to\infty}\left(\frac{x^2+2x+5}{2x-1} - \frac{x}{2}\right)$$
$$= \lim_{x\to\infty} \frac{2(x^2+2x+5) - x(2x-1)}{2(2x-1)}$$
$$= \lim_{x\to\infty} \frac{5x+10}{4x-2} = \frac{5}{4}$$

Conclusion: the oblique asymptote is the line: $y = \dfrac{x}{2} + \dfrac{5}{4}$.

Graph of $f(x) = \dfrac{x^2+2x+5}{2x-1}$



Example. A nation's consumption function is given by

$$C = \frac{9Y^2 + 10Y + 15}{10Y + 1}$$

where consumption C and income Y are both measured in billions of dollars.

Somewhat vague question: *How does the consumption function behave when income is very large?*

One answer: See if there is an oblique asymptote.

$$\begin{aligned}\frac{dC}{dY} &= \frac{(18Y + 10)(10Y + 1) - 10(9Y^2 + 10Y + 15)}{(10Y + 1)^2} \\ &= \frac{90Y^2 - 140}{100Y^2 + 20Y + 1}\end{aligned}$$

so

$$\lim_{Y \to \infty} \frac{dC}{dY} = \lim_{Y \to \infty} \frac{90Y^2 - 140}{100Y^2 + 20Y + 1} = 0.9.$$

$\Rightarrow$ There is an oblique asymptote with slope 0.9....

Next:

$$\begin{aligned}\lim_{Y\to\infty}(C-0.9Y) &= \lim_{Y\to\infty}\left(\frac{9Y^2+10Y+15}{10Y+1}-0.9Y\right)\\ &= \lim_{Y\to\infty}\left(\frac{9Y^2+10Y+15-9Y^2-0.9Y}{10Y+1}\right)\\ &= \lim_{Y\to\infty}\left(\frac{9.1Y+15}{10Y+1}\right)=\frac{9.1}{10}=0.91\end{aligned}$$

Conclusion: The line $\tilde{C}=0.9Y+0.91$ is an oblique asymptote to the consumption curve. I.e., when income is large, the consumption 'behaves like' the linear function $\tilde{C}=0.9Y+0.91$.