Limits at infinity of rational functions.

Observation: If x is very large, then

$$a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0 = x^m \cdot \overbrace{(a_m + a_{m-1}x^{-1} + \cdots + a_0 x^{-m})}^{=a_m+\text{tiny}} \approx a_m x^m$$

and

$$b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 = x^n \cdot \overbrace{(b_n + b_{-1}x^{n-1} + \cdots + b_0 x^{-n})}^{=b_n+\text{tiny}} \approx b_n x^n$$

This means that if x is very large, then

$$\frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0} \approx \frac{a_m x^m}{b_n x^n} = \frac{a_m}{b_n} x^{m-n}.$$

Therefore,

$$\lim_{x\to\infty} \frac{a_m x^m + \cdots + a_0}{b_n x^n + \cdots + b_0} = \lim_{x\to\infty} \frac{a_m}{b_n} x^{m-n} = \begin{cases} 0 & : \quad m < n \\ a_n/b_n & : \quad m = n \\ \pm\infty & : \quad m > n \end{cases}$$

Examples:

$$\lim_{x\to\infty} \frac{1+2x-3x^2}{4x+5} = -\infty$$

because $2 > 1$ and $-3/4 < 0$.

$$\lim_{x\to\infty} \frac{\pi x^3 - 2x^2 + 300x - 7}{2x^3 + 55} = \frac{\pi}{2},$$

because $3 = 3$.

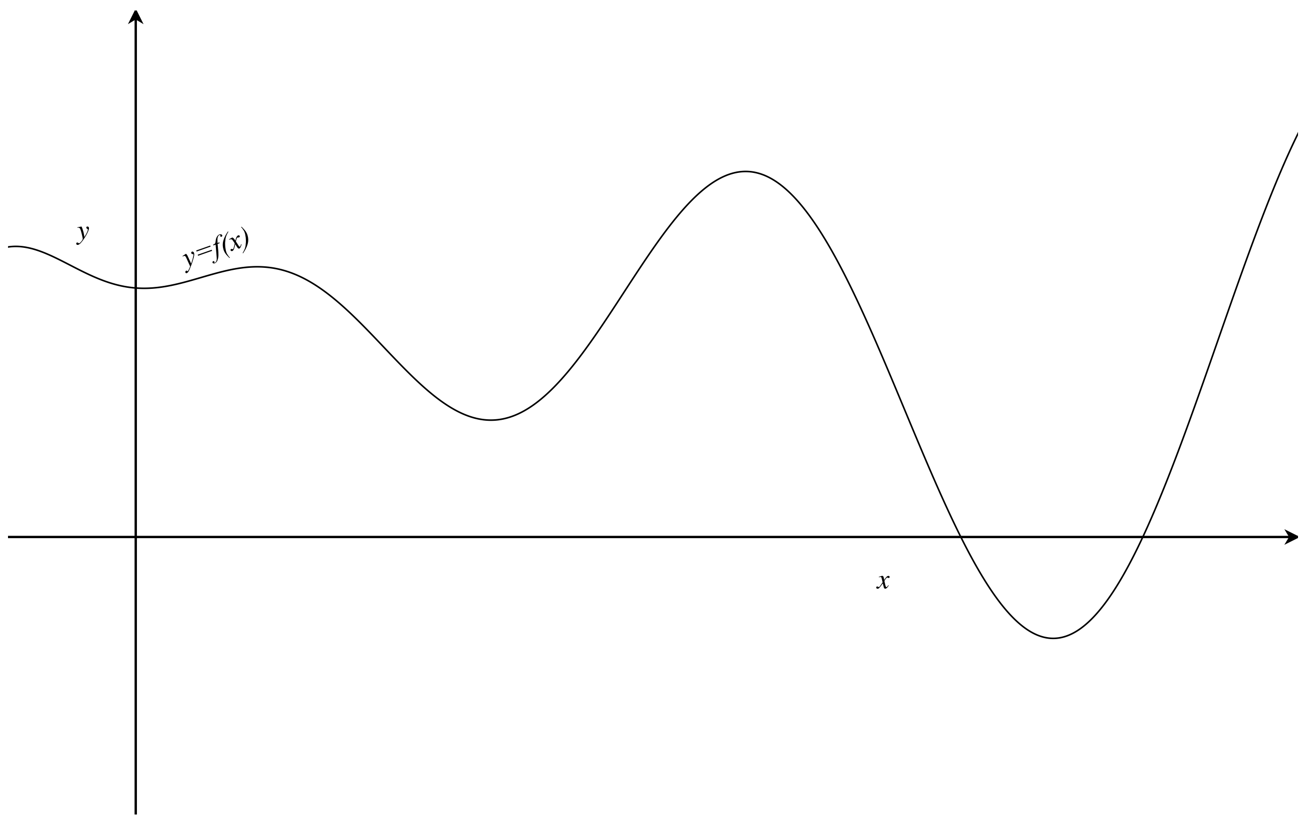
$$\lim_{u\to\infty} \frac{50u^2 + 100u + 2000}{0.01u^3 + 1} = 0,$$

because $2 < 3$.

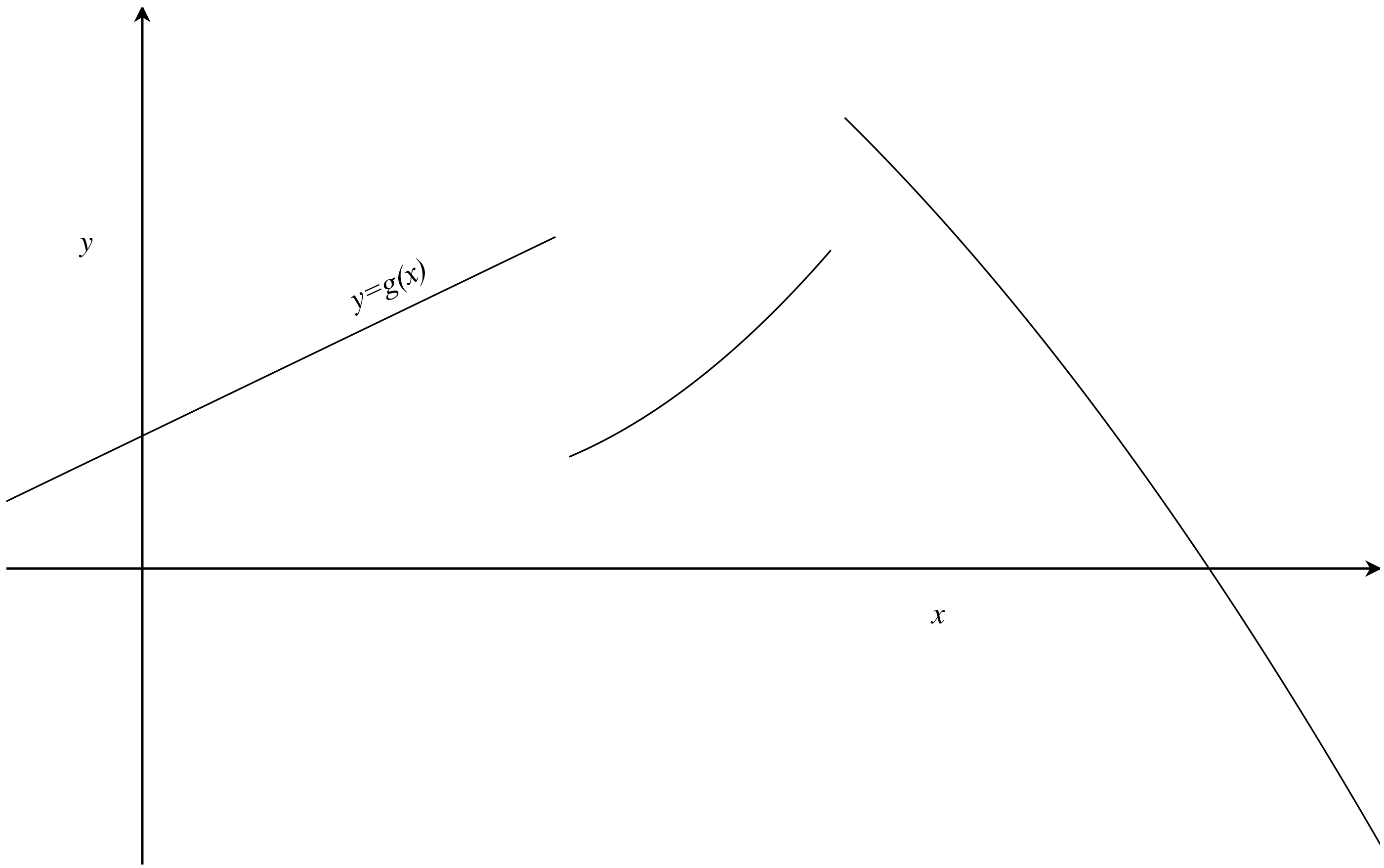
Continuity

Intuition: *The function $f(x)$ is **continuous** in the interval (a, b) if the graph $y = f(x)$ is an unbroken (i.e., continuous) curve over that interval.*

The graph of a continuous function:



The graph of a function that is *not* continuous (at some points):



Observation: The 'unbroken graph' intuition about continuity is useful for the visual guidance it provides, but it is not useful for understanding the properties of continuous functions.

For that, we need a more precise definition...

Definitions:

A. *The function $f(x)$ is* ***continuous at the point*** $\boldsymbol{x = x_0}$ *if*

(i) $f(x)$ is defined at $x = x_0$, and

(ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

B. *The function $f(x)$ is* ***continuous in the interval I=(a,b)*** *if $f(x)$ is continuous at every point x_0 in I.*

Basic continuous functions:

1. Constant functions are continuous for all real x

Because if $f(x) = C$ for all x and x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} C = C = f(x_0).$$

2. If k is a positive integer, then $f(x) = x^k$ is continuous for all real x

Because if x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^k = x_0^k = f(x_0).$$

3. If α is any real number, then $f(x) = x^\alpha$ is continuous for all $x > 0$

Because if x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^\alpha = x_0^\alpha = f(x_0).$$

4. The function $f(x) = e^x$ is continuous for all x.

I.e., if x_0 is any real number, then

$$\lim_{x \to x_0} e^x = e^{x_0}.$$

5. The function $f(x) = \ln x$ is continuous for all $x > 0$.

I.e., if x_0 is any *positive* real number, then

$$\lim_{x \to x_0} \ln x = \ln x_0.$$

Combinations:

If $f(x)$ and $g(x)$ are both continuous, then

6. $f(x) + g(x)$ and $f(x) - g(x)$ are both continuous,

7. $f(x)g(x)$ is continuous and

8. $f(x)/g(x)$ is continuous at every point where $g(x) \neq 0$.

Composite functions:

9. If $g(x)$ is continuous in the interval I and $f(x)$ is continuous in

$$g(I) = \{g(x) : x \in I\}$$

then $f(g(x))$ is continuous in I.

Examples:

(a) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial, then $f(x)$ is continuous for all x because

(i) x^k is continuous for all x if k is a positive integer;

(ii) a_k is continuous for each k, since it is constant;

(iii) $a_k x^k$ is continuous, as a product of continuous functions;

(iv) $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous as a sum of continuous functions.

(b) If $Q(x) = \dfrac{f(x)}{g(x)}$ is a rational function (i.e., $f(x)$ and $g(x)$ are both polynomials), then $Q(x)$ is continuous at all points where $g(x) \neq 0$.

(c) The function $f(x) = \sqrt{x}$ is continuous for all $x > 0$ and $g(x) = x^2+2$ is continuous for all x. Furthermore, $g(x) \geq 2 > 0$ for all x, so $f(g(x)) = \sqrt{x^2+2}$ is continuous for all x.

(d) The function $|x|$ is continuous for all x. Because...

$$|x| = \begin{cases} x & : \quad x \geq 0 \\ -x & : \quad x < 0 \end{cases}$$

so $|x|$ is continuous for $x > 0$ and for $x < 0$ and it just remains to show that $|x|$ is continuous at $x = 0$. , which we do by considering 1-sided limits:

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \quad \text{and} \quad \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0,$$

so $\lim_{x\to 0} |x| = 0 = |0|$ so the function $|x|$ is also continuous at $x = 0$.

(e) The function $\ln x$ is continuous for all $x > 0$, $|x|$ is continuous for all x and $|x| > 0$ for all $x \neq 0$, so the function $\ln |x|$ is continuous for all $x \neq 0$.

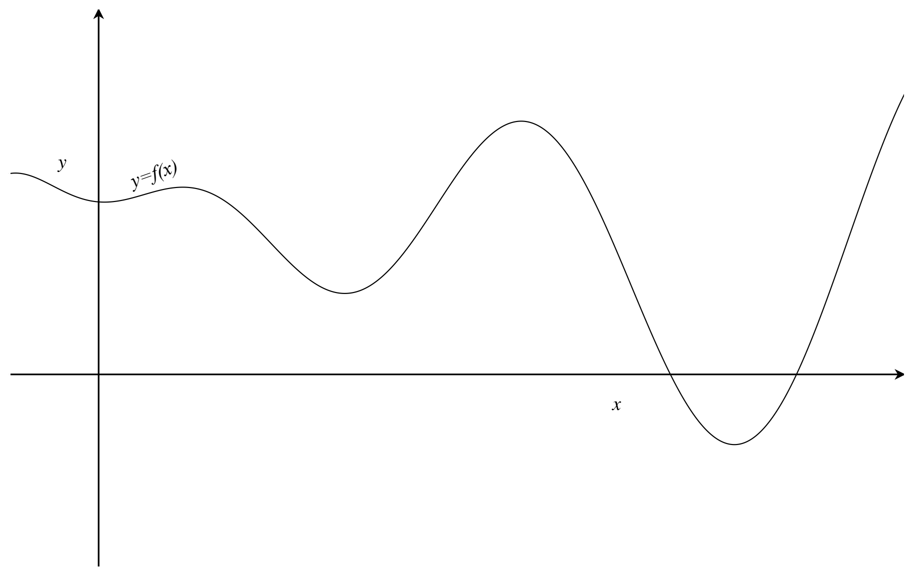
(f) **Question:** on what interval(s) is the function

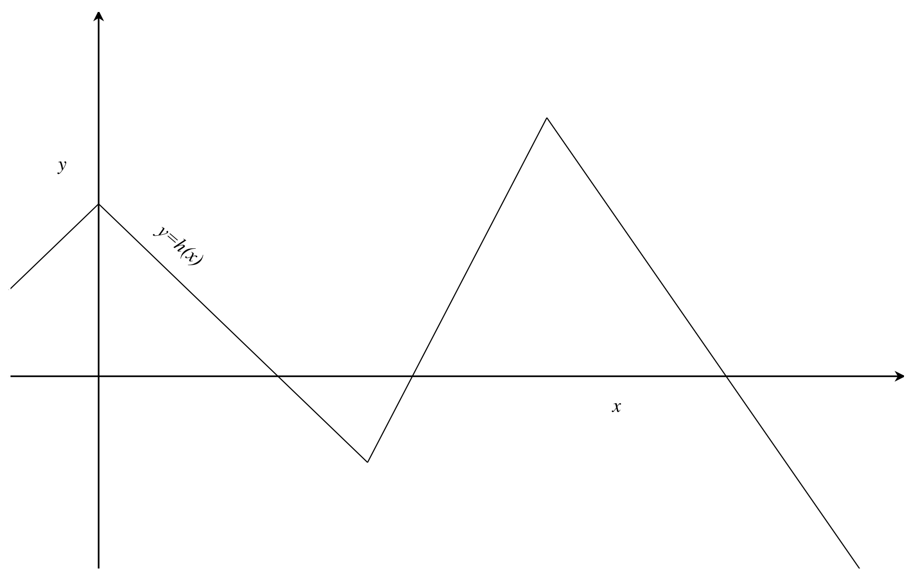
$$h(x) = \sqrt{x^2 - x - 2}$$

continuous?

Answer: The function $f(x) = \sqrt{x}$ is continuous for all $x > 0$, and the function $g(x) = x^2 - x - 2$ is continuous for all x, so $h(x) = f(g(x))$ will be continuous at all points where $g(x) > 0$. Now $g(x) = x^2 - x - 2 = (x+1)(x-2)$, so $g(x) > 0$ if both factors are positive or if both factors are negative. Both factors are positive when $x > 2$ and both factors are negative when $x < -1$, so $h(x)$ is continuous on the intervals $(-\infty, -1)$ and $(2, \infty)$.

Continuous vs. Smooth



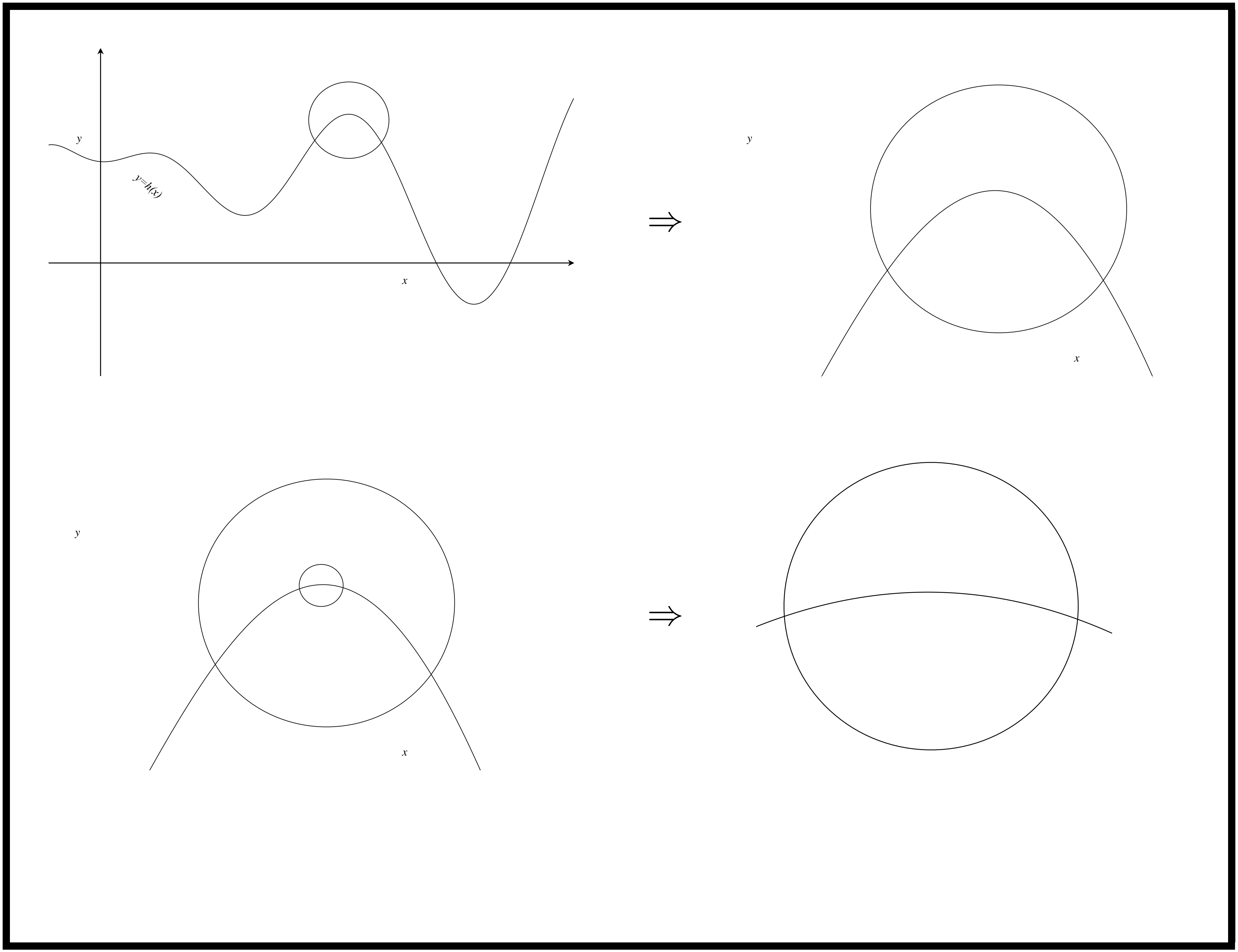


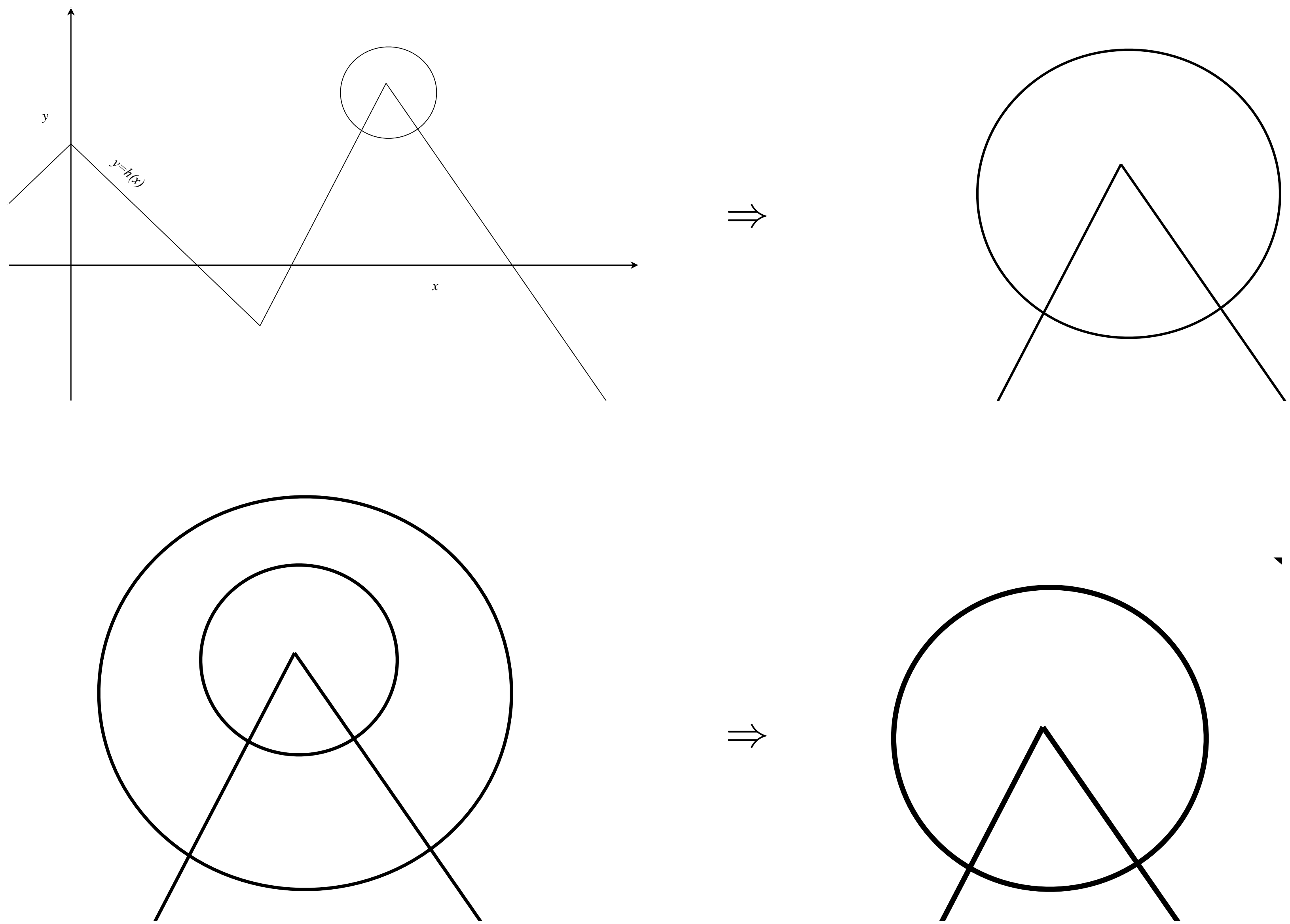
Differences:

1. The ‘smooth graph’ changes directions gradually, while the ‘sawtooth’ graph changes directions abruptly, without warning.

2. By zooming in, it seems that we can associate a specific direction, or slope to every point on the ‘smooth’ graph, but at the points where the ‘sawtooth’ graph changes direction, there is no well-defined direction.

3. In fact, zooming in on the ‘smooth’ graph makes the graph flatter (straighter) around each point. When we zoom in on the corners of the sawtooth graph, the corner never goes away — the graph does not become flat there.


y
y=h(x)
x
⇒
y
x
y
x
⇒


y
y=h(x)
x
⇒
⇒

Observation: As we zoom in to a point on the smooth graph, the graph becomes more and more like a straight line there. The direction (slope) of that (hypothetical) straight line is what we call the *slope of the graph* at that point.

Definition: *The* ***tangent line*** *to the graph* $y = f(x)$ *at the point* $(x_0, f(x_0))$, *is the line that passes through the point* $(x_0, f(x_0))$ *and has the same slope as the graph at that point.*

Question: How do we know that the graph has a slope, and if it does, how do we find it?

First step: Give a precise *definition* for what we mean by slope.

Recall that the slope of a *straight line* is defined by the *difference quotient*

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

where (x_0, y_0) and (x_1, y_1) are points on the line satisfying $x_1 \neq x_0$.

(*) If no such points exist on the line, then the line is vertical and technically does not have a slope.

The problem we encounter when we try to define the slope of $y = f(x)$ at the point $(x_0, f(x_0))$ is that we only have one point to work with.

The solution to this problem, is to ...

(i) ... find an approximate value for the slope at the given point,

(ii) find a way to improve the approximation and

(iii) compute the limit of the (improving) approximations. This limit, *if it exists*, will be the slope we seek.