Percentage Change and Elasticity

1. Relative and percentage rates of change

The derivative of a differentiable function $y = f(x)$ describes how the function changes. The value of the derivative at a point, $f'(x_0)$, gives the *instantaneous rate of change* of the function at that point, which we can understand in more practical terms via the *approximation formula*

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x,$$

which is accurate when Δx is sufficiently small, (see SN5 for more details). This formula describes the change Δy in terms of the change Δx, which is very useful in a variety of contexts. In other contexts however, it is sometimes more appropriate to describe the change in y in *relative* terms. In other words, to compare the change in y to its previous value.

Suppose, for example that you learn that a firm's revenue increased by \$2.5 million over the past year. An increase in revenue is generally a good thing, but how good is a \$2.5 million increase? The answer depends on the size of the firm, and more specifically, on the amount of last year's revenue. If in the previous year the firm's revenue was \$2.1 million, then an increase of \$2.5 million means that revenue more than doubled, which means that the firm's business is growing very robustly. On the other hand, if last year's revenue was \$350 million, then \$2.5 million is an increase of less than 1%, which is far less impressive.

The *relative change* in the value of a variable or function is simply the ratio of the change in value to the starting value. I.e., the relative change in $y = f(x)$ is given by

$$\frac{\Delta y}{y} = \frac{f(x_0 + \Delta x) - f(x_0)}{f(x_0)}.$$

Likewise, the *relative rate of change* (rroc) of a function $y = f(x)$ is obtained by dividing the derivative of the function by the function itself,

(1) $$\text{rroc} = \frac{f'(x)}{f(x)} = \frac{1}{y} \cdot \frac{dy}{dx}.$$

Example 1. Suppose that $y = x^2 + 1$, then $\dfrac{dy}{dx} = 2x$, and the relative rate of change of y with respect to x is

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2x}{x^2 + 1}.$$

When $x = 4$, the relative rate of change is $2 \cdot 4/(4^2 + 1) = 8/17 \approx 0.47$, and when $x = 10$, the relative rate of change is $2 \cdot 10/(10^2 + 1) = 20/101 \approx 0.198$.

The relative change, and the relative rate of change are often expressed in the percentage terms, in which case they are usually called the *percentage change* and the *percentage rate of change*, respectively. Thus, the percentage change in y is given by

$$\%\Delta y = \frac{\Delta y}{y} \cdot 100\% = \frac{f(x_0 + \Delta x) - f(x_0)}{f(x_0)} \cdot 100\%,$$

and the percentage rate of change (%-roc) of the function $y = f(x)$ is simply the relative rate of change multiplied by 100%,

$$\text{\%-roc} = \text{rroc} \cdot 100\% = \frac{f'(x)}{f(x)} \cdot 100\% = \frac{1}{y} \cdot \frac{dy}{dx} \cdot 100\%.$$

Example 1. (*continued*) The percentage rate of change of the function $y = x^2 + 1$ is $(1/y) \cdot (dy/dx) \cdot 100\% = \dfrac{2x}{x^2 + 1} \cdot 100\%$. When $x = 4$, the percentage rate of change of the function is

$$\frac{8}{17} \cdot 100\% \approx 47\%,$$

and when $x = 10$ the percentage rate of change is

$$\frac{20}{101} \cdot 100\% \approx 19.8\%.$$

Since the approximation formula provides an estimate for Δy, it can also be used to estimate the percentage change, $(\Delta y/y) \cdot 100\%$. Specifically, if Δx is sufficiently small, then the approximation formula says that $\Delta y \approx f'(x_0) \cdot \Delta x$, and therefore

$$\%\Delta y = \frac{\Delta y}{y} \cdot 100\% \approx \frac{f'(x_0) \cdot \Delta x}{f(x_0)} \cdot 100\%.$$

If we rearrange the factors on the right-hand side we obtain

Fact 1. *If $y = f(x)$ is differentiable at the point $x = x_0$ and Δx is sufficiently small, then*

$$(2) \qquad \%\Delta y = \frac{f(x_0 + \Delta x) - f(x_0)}{f(x_0)} \cdot 100\% \approx \left(\frac{f'(x_0)}{f(x_0)} \cdot 100\% \right) \cdot \Delta x.$$

In words, if Δx is sufficiently small, then the *percentage change* in y is approximately equal to the *percentage **rate** of change* of y multiplied by the change in x.

Example 1. (*conclusion*)   We have already seen that for $y = x^2 + 1$, the percentage rate of change when $x = 10$ is $20/101 \approx 19.8\%$. Now, suppose that x increases from $x_0 = 10$ to $x_1 = 10.4$, so that $\Delta x = 0.4$. Then, according to equation (2), the percentage change in y will be approximately

$$\%\Delta y \approx (19.8\%) \cdot (0.4) = 7.92\%.$$

2. Elasticity

Economists frequently describe changes in percentage terms, and as described above, the derivative can be used to compute the percentage rate of change of a function. Another measure of change that we commonly see in economics is *elasticity*. Average elasticity is defined as a ratio of percentage changes.

Definition 1.   For the function $y = f(x)$, the ***average elasticity***, $E_{y/x}$, of the variable y with respect to the variable x is given by the ratio

$$(1) \qquad E_{y/x} = \frac{\%\Delta y}{\%\Delta x} = \frac{(\Delta y/y) \cdot 100\%}{(\Delta x/x) \cdot 100\%} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y}.$$

Note that elasticity is a ***units-free measure*** of change (a *pure number*), because the units for x are canceled in the ratio $x/\Delta x$ and the units for y are canceled in the ratio $\Delta y/y$.

Example 2. A firm's short-term production function is given $Q = 20L^{1/4}$, where Q is weekly output, measured in \$1000's and L is labor input, measured in 100's of weekly worker-hours. In preparation for the holiday season, the firm increases labor input from 4000 hours per week to 4100 hours per week, in other words, L increases from $L_0 = 40$ to $L_1 = 41$. Correspondingly, the firm's output increases from $Q_0 = 20 \cdot 40^{1/4} \approx 50.297$, to $Q_1 = 20 \cdot 41^{1/4} \approx 50.609$.

In percentage terms, the change in labor input is $\%\Delta L = (1/40) \cdot 100\% = 2.5\%$, and the change in output is $\%\Delta Q \approx (0.311/50.297) \cdot 100\% \approx 0.619\%$. The firm's average *labor elasticity of output* is therefore

$$E_{Q/L} = \frac{\%\Delta Q}{\%\Delta L} \approx \frac{0.619\%}{2.5\%} \approx 0.2476.$$

If we compute the average elasticity over shorter and shorter x-intervals, i.e., we let Δx go to 0, we are lead to the concept of *point elasticity*.

Definition 2. *For the function $y = f(x)$, the **point elasticity**, $\eta_{y/x}$, of y with respect to x is the limit of the average elasticity as $\Delta x \to 0$:*

$$(2) \qquad \eta_{y/x} = \lim_{\Delta x \to 0} E_{y/x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right) = \frac{dy}{dx} \cdot \frac{x}{y}.$$

It is common to omit the word 'point' when referring to point elasticity, and simply speak of the elasticity of y with respect to x. In fact, typically we refer to $\eta_{y/x}$ as '*the x-elasticity of y*.' For example, the elasticity of output with respect to labor input is called *labor-elasticity of output*, and the elasticity of the demand for a good with respect to the good's price is called *price-elasticity of demand*.

Example 3. Let's compute the labor-elasticity of output for the production function in Example 0.0.2. First we compute the derivative,

$$\frac{dQ}{dL} = \frac{1}{4} \cdot 20L^{-3/4} = 5L^{-3/4},$$

then the elasticity is given by

$$\eta_{Q/L} = \frac{dQ}{dL} \cdot \frac{L}{Q} = 5L^{-3/4} \cdot \frac{L}{20L^{1/4}} = \frac{1}{4}.$$

So, the elasticity is *constant* in this case. Note also that the average elasticity that we computed in Example 0.0.2 is fairly close to the point elasticity. This should not be a surprise, since point elasticity is the limit of average elasticity as the change in the free variable approaches 0.

Example 4. The supply function for the widget market is given by

$$q_s(p) = 0.01p^2 + 12p - 50,$$

where $q_s(p)$ is the quantity of widgets that the widget industry will supply (per unit time) when the price per widget is p. Let's compute the *price-elasticity of supply* for this market. From Definition 2 we have

$$\eta_{q_s/p} = \frac{dq_s}{dp} \cdot \frac{p}{q} = (0.02p + 12) \cdot \frac{p}{0.01p^2 + 12p - 50} = \frac{0.02p^2 + 12p}{0.01p^2 + 12p - 50}.$$

You'll notice that $\eta_{q_s/p}$ is a function of p, which is not surprising given its definition. In other words, normally we should expect the elasticity to change when the variable changes, so functions that exhibit *constant elasticity*, as in the previous example, are fairly unique.

To compute the price-elasticity of supply when the price is $p_1 = \$10$, we simply plug $p = 10$ into the formula that we computed above

$$\eta_{q_s/p}\Big|_{p=10} = \frac{0.02 \cdot 100 + 12 \cdot 10}{0.01 \cdot 100 + 12 \cdot 10 - 50} = \frac{122}{71} \approx 1.71833.$$

Likewise, the price-elasticity of supply when the price is $p_2 = 20$ is

$$\eta_{q_s/p}\Big|_{p=20} = \frac{0.02 \cdot 400 + 12 \cdot 20}{0.01 \cdot 400 + 12 \cdot 20 - 50} = \frac{248}{194} \approx 1.27835.$$

Question: what happens to price elasticity of supply as the price grows very large?

How can elasticity be interpreted? E.g., what does it mean that $\eta_{q_s/p}\Big|_{p=10} \approx 1.71833$? What are we to make of that number? One answer is given by the *approximation formula for elasticity.*

Fact 2. *If Δx is sufficiently small, then*

(3)
$$\boxed{\%\Delta y \approx \eta_{y/x} \cdot \%\Delta x}$$

This approximation formula follows directly from the definition of (point) elasticity, namely

$$\eta_{y/x} = \lim_{\Delta x \to 0} E_{y/x} = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x}.$$

From this it follows that if Δx is sufficiently small (close to 0), then

$$\eta_{y/x} \approx \frac{\%\Delta y}{\%\Delta x},$$

and multiplying both sides of this (approximate) identity by $\%\Delta x$ gives the approximation formula (3), above.

Example 4. (*continued*) Using the approximation formula for elasticity, we can interpret the numbers that we computed above. E.g., we saw that $\eta_{q_s/p}\Big|_{p=10} \approx 1.71833$, what does this tell us about the change in supply of widgets, if the price increases by 5%?

First of all, if the price increases by 5% from $p_1 = \$10$, this corresponds to $\Delta p = 0.5$, which is small enough for our purposes, so we invoke the approximation formula. According to that formula, if the price increases by 5%, then the percentage change in supply will be

$$\%\Delta q_s \approx \eta_{q_s/p}\Big|_{p=10} \cdot 5\% \approx 1.71833 \cdot 5\% = 8.59165\%.$$

Similarly, if the price is increased by 5% from $p_2 = 20$, then the percentage change in supply will be $\%\Delta q_s \approx \eta_{q_s/p}\Big|_{p=10} \cdot 5\% \approx 1.27835 \cdot 5\% = 6.39175\%.$

Question: If the price of widgets is very high, and is then changed by 1%, what will the (approximate) effect on supply be?

3. *Price elasticity of demand*[†]

Suppose that the demand function for a (monopolistic) firm is given by $q = g(p)$. Then the price-elasticity of demand for this firm's product is given by Equation (2), namely

(1)
$$\eta_{q/p} = \frac{dq}{dp} \cdot \frac{p}{q}.$$

[†]This material is also covered in the book in section 12.3. You should read that section in detail.

In this case, as expected, the price-elasticity of demand appears as a function of the price, i.e.,

$$\eta_{q/p} = \frac{g'(p)}{g(p)} \cdot p.$$

However, when it comes to demand functions, they are often given with price as the *dependent* variable and demand as the *independent* variable. In other words, very often demand functions look like this: $p = f(q)$. This form notwithstanding, economists still want to find the price-elasticity of demand, and this presents a small problem, but one that is very easily solved.

The problem is this. To compute $\eta_{q/p}$, we need to compute dq/dp, but if the relationship between price and demand is in the form $p = f(q)$, then it may not be immediately obvious how to compute the derivative of q with respect to p. The solution to this problem is provided by the following important fact.

Fact 3. *If $y = f(x)$ and $\dfrac{dy}{dx} \neq 0$, then*

(2)
$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.$$

In the case of demand functions, this identity means that

$$\frac{dq}{dp} = \frac{1}{\dfrac{dp}{dq}},$$

(as long as $dp/dq \neq 0$). This allows us to compute the price-elasticity of demand in the case that the demand function appears in the form $p = f(q)$. Specifically, in this case, we find that $\eta_{q/p}$ is given by

(3)
$$\eta_{q/p} = \frac{dq}{dp} \cdot \frac{p}{q} = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}}.$$

You should note that in this case, price-elasticity of demand is given as a function of the variable q. I.e., if the demand function is $p = f(q)$, then

$$\eta_{q/p} = \frac{f(q)}{q \cdot f'(q)}.$$

Example 5. Suppose that the demand function for a monopolist's product is given by

$$p = \sqrt{100 - 0.06q^2}.$$

Find the price-elasticity of demand when $p = 5$.

In this example, the demand function has the form $p = f(q)$, so we need to use Equation (3) to compute the price-elasticity of demand. First, compute dp/dq,

$$\frac{dp}{dq} = \frac{1}{2}(100 - 0.06q^2)^{-1/2} \cdot (-0.12q) = -\frac{0.06q}{\sqrt{100 - 0.06q^2}}.$$

Now, plug this into (3)

$$\eta_{q/p} = \frac{p/q}{dp/dq} = \frac{\sqrt{100 - 0.06q^2}/q}{-0.06q/\sqrt{100 - 0.06q^2}} = -\frac{100 - 0.06q^2}{0.06q^2}.$$

Note that $\eta_{q/p}$ is expressed as a function of q in this case, so to find the price-elasticity of demand when $p = 5$, we need to find the value of $q^{\ddagger}$ that corresponds to this price. In other words we need to solve the equation $\sqrt{100 - 0.06q^2} = 5$ for q.

$$\sqrt{100 - 0.06q^2} = 5 \implies 100 - 0.06q^2 = 25 \implies 0.06q^2 = 75 \implies q^2 = 1250.$$

This means that $q = \sqrt{1250}$, since demand must be positive. Finally, we see that

$$\left.\eta_{q/p}\right|_{p=5} = \left.\eta_{q/p}\right|_{q=\sqrt{1250}} = -\frac{100 - 0.06 \cdot 1250}{0.06 \cdot 1250} = -\frac{1}{3}.$$

The approximation formula for elasticity (Equation (3)) may now be used to predict the change in demand that corresponds to different changes in the price. For example, if the price increases from \$5 to \$5.25, then in percentage terms the change in price is $\%\Delta p = 5\%$, so the percentage change in demand will be

$$\%\Delta q \approx \left.\eta_{q/p}\right|_{p=5} \cdot \%\Delta p = -\frac{1}{3} \cdot 5\% \approx -1.667\%.$$

Comments:

1. Price-elasticity of demand, as defined here (and in the book), is always negative. This reflects the fact that if price increases, then demand decreases, and vice versa. Many economists, (and books), define elasticity to be a positive quantity. Thus, in the case of price-elasticity of demand for example, it is defined to be the *absolute value* of the quantity defined here.

2. If $|\eta_{q/p}| > 1$, then a 1% change in price will result in a *greater-than-1%* change in demand, as follows from the approximation formula. In this case, demand is said to be **elastic**.

3. Likewise, If $|\eta_{q/p}| < 1$, then a 1% change in price will result in a *smaller-than-1%* change in demand. In this case, demand is said to be **inelastic**.

4. If $|\eta_{q/p}| = 1$ (i.e., $\eta_{q/p} = -1$), then demand is said to have **unit elasticity**. In this case, small percentage changes in price result in the same percentage change in demand, in the opposite direction.

3.1 *Price-elasticity of demand and marginal revenue.*

There is a simple relationship between price-elasticity of demand and marginal revenue. To see this relationship, we begin by writing $r = p \cdot q$, and differentiating both sides with respect to q, which gives

$$\frac{dr}{dq} = \frac{dp}{dq} \cdot q + p \cdot \frac{dq}{dq} = \frac{dp}{dq} \cdot q + p,$$

using the product rule to differentiate the right-hand side.

Next, we pull a factor of p out of both terms on the right-hand side, above, to obtain

$$\frac{dr}{dq} = p \left(\frac{dp}{dq} \cdot \frac{q}{p} + 1 \right).$$

Finally, we observe that

$$\frac{dp}{dq} \cdot \frac{q}{p} = \frac{dp/dq}{p/q} = \frac{1}{\eta_{q/p}},$$

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‡Looking at the form of the elasticity function in this case, you may notice that it is enough to find the value of $0.06q^2$ that corresponds to $p = 5$.

and conclude that

(4)
$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta_{q/p}}\right).$$

From this simple identity it follows that

- If demand is elastic then marginal revenue is positive, because $\eta_{q/p} < -1$ in this case, so $-1 < 1/\eta_{q/p} < 0$, and

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta_{q/p}}\right) > 0,$$

  since price is always positive.
- If demand is inelastic then marginal revenue is negative, because $0 > \eta_{q/p} > -1$ in this case, so $1/\eta_{q/p} < -1$, and

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta_{q/p}}\right) < 0.$$

- If demand has unit elasticity, then marginal revenue is equal to 0 (prove it!).

As we will see in the chapter on optimization, the last observation implies that (under suitable assumptions) *revenue is maximized at the price for which demand has unit elasticity.*

Example 6. The price elasticity of demand for ACME Widgets is give by

$$\eta_{q/p} = \frac{-3p}{p^2 + 1.25}.$$

What will happen to ACME's revenue if they raise the price of their product from it's current level, $p_0 = 1$, to $p_1 = 1.15$?

The price-elasticity of demand when $p_0 = 1$ is

$$\eta_{q/p}\Big|_{p=1} = \frac{-3}{2.25} = -\frac{4}{3}.$$

From the relation between marginal revenue and elasticity, (4), it follows that

$$\frac{dr}{dq}\Big|_{p=1} = 1 \cdot \left(1 + \frac{1}{\eta_{q/p}\big|_{p=1}}\right) = 1 - \frac{3}{4} = \frac{1}{4} > 0.$$

So marginal revenue is positive.

Now, recall that if the price goes up, then the demand goes down. This means that, while we don't know the precise value of Δq, we can say that if $\Delta p = 0.15 > 0$, then $\Delta q < 0$. Finally, using the approximation formula for the revenue function, we have

$$\Delta r \approx \frac{dr}{dq}\Big|_{p=1} \cdot \Delta q = \frac{1}{4} \cdot \Delta q < 0.$$

So we can conclude that if the firm raises their price from $p_0 = 1$ to $p_1 = 1.14$, then their revenue will decrease.

Exercises.

1. A firm's production function is given by $Q = L^{4/3} + 25L$, where L is labor input. Find the firm's labor-elasticity of output when $L_0 = 27$, and use your answer to find the approximate percentage change in output if labor input is increased to $L_1 = 28.5$.

2. The function $q = 20\ln(y^{3/2} + y + 1)$ describes the (weekly) demand for widgets, q, in Mudville, as it depends on the average (weekly) household income there, y.

(a) Find the income-elasticity of demand for widgets in Mudville as a function of y.

(b) What is the income-elasticity of demand when the average household income is $500.00? (Round your answer to 2 decimal places.)

(c) What will be the approximate percentage change in demand for widgets, if average household income increases from $500.00 to $510.00?

3. The demand function for a monopolistic firm is $p = 250 - q - 0.02q^2$.

(a) Find the price-elasticity of demand when $q = 80$. Is demand elastic or inelastic at this point?

(b) Find the **price** for which demand has unit elasticity.

4. The demand function for a (different) firm's product is $q = 3000 - 16p^{5/4}$.

(a) Find the firm's price-elasticity of demand, **as a function of** $\boldsymbol{p}$.

(b) Use your answer to (a) to calculate the (approximate) percentage change in demand when the firm lowers the price of its product from $p_0 = 16.00$ to $p_1 = 15.00$.

(c) How will this price change affect the firm's revenue? Use your answer to (a) to justify your answer.