Marginal Functions and Approximation

1. *Linear approximation*

If $y = f(x)$ is a differentiable function then its derivative, $y' = f'(x)$, gives the *rate of change* of the variable y with respect to the variable x. The term 'rate of change' comes from the definition of the derivative

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \tag{1}$$

where $\Delta x = x - x_0$, is the change in the value of the variable x, and $\Delta y = f(x_0 + \Delta x) - f(x_0)$, is the corresponding change in the value of the variable y. Thus, the derivative is the limit, (as Δx goes to 0), of the *ratio of the changes* $\Delta y/\Delta x$. So rate-of-change comes from ratio-of-changes.

The notion of rate-of-change can be made more concrete by remembering the definition of the *limit*. Specifically, if Δx is sufficiently close to 0, then the ratio on the right-hand side of equation (1) is approximately equal to the value of the derivative on the left-hand side. I.e., if Δx is small, then

$$\frac{\Delta y}{\Delta x} \approx f'(x_0). \tag{2}$$

Now, if we multiply both sides of the approximate equality above by Δx we obtain the simple, but very important *linear approximation* formula.

Fact 1. *If $y = f(x)$ is differentiable at $x = x_0$, and if Δx is sufficiently small, then*

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x. \tag{3}$$

Comments:

a. If $|\Delta x| < 1$, then the approximation in (3) is more accurate than the one in (2). (Can you say why?)
b. The quality of the estimate given by the approximation formula depends very strongly on the specific function $f(x)$, the point x_0, and the size of Δx. I will illustrate this dependence in the examples of Section 2, below.

The linear approximation formula (and its name) can be explained geometrically. Remember that the derivative $f'(x_0)$ may also be interpreted as the *slope of the tangent line* to the graph $y = f(x)$ at the point $(x_0, f(x_0))$. Using the *point-slope* formula, we can find that the equation of this tangent line is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The tangent line can also be considered to be the graph of the linear function

$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$

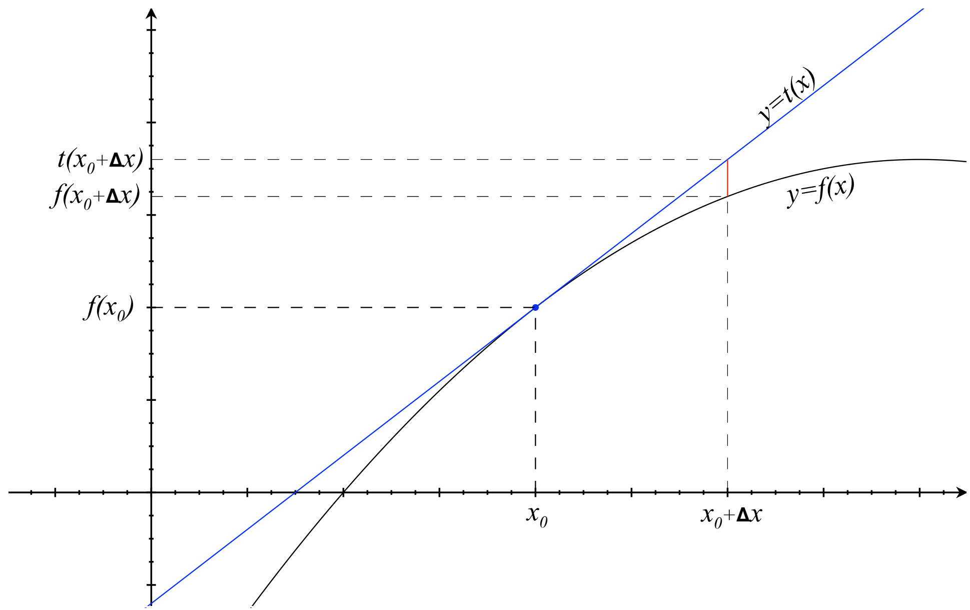

FIGURE 1. Graphical representation of linear approximation.

and the difference $\big|\, f(x_0+\Delta x) - t(x_0+\Delta x)\big|$ is the (vertical) *distance* between the tangent line and the graph $y = f(x)$ at the point $x_0 + \Delta x$. This difference simplifies to

$$\begin{aligned}\Big|\, f(x_0+\Delta x) - t(x_0+\Delta x)\Big| &= \Big|\, f(x_0+\Delta x) - \big[\, f(x_0) + f'(x_0)((x_0+\Delta x) - x_0)\big]\Big| \\ &= \Big|\, \big[\, f(x_0+\Delta x) - f(x_0)\big] - f'(x_0)\cdot\Delta x\Big| \\ &= \big|\,\Delta y - f'(x_0)\cdot\Delta x\big|\,.\end{aligned}$$

Geometrically speaking, linear approximation says that if $\Delta x = x - x_0$ is sufficiently small, then the *linear function* $y = t(x)$ approximates the original function $y = f(x)$ quite well, as illustrated in Figure 1.

2. *Examples of linear approximation in action*

This section contains simple examples that illustrate how the accuracy of the approximation in *linear approximation* depends on the function $f(x)$, the starting point x_0 and the change in the x-variable, Δx. You can skip this subsection if you want, but I recommend that you don't. In fact I recommend that you redo all the computations that I do below for good measure.

Example 1. Suppose that $f(x) = \sqrt{x} = x^{1/2}$, so $f'(x) = \frac{1}{2}x^{-1/2}$. I'll use linear approximation with this function and two values of x_0 and several different values of Δx. First, if $x_0 = 4$, then linear approximation reads

$$\Delta y = \sqrt{4+\Delta x} - \sqrt{x} \approx \frac{1}{2}\cdot 4^{-1/2}\cdot\Delta x = \frac{\Delta x}{4}.$$

Now, I'll compare the estimated change in the y-value provided by the approximation above, to the actual[1] change in the y-value for several values of Δx. For neatness' sake, I've collected the results in the table below, in which the left-hand column contains the different

[1] Note that this 'actual' difference is also an approximation, albeit a much more accurate one.

values of Δx, the middle column gives the corresponding *estimates* for Δy provided by the approximation formula and the right-hand column gives the *actual* values (rounded to 5 decimal places) of Δy.

Δx	$f'(x_0) \cdot \Delta x$	Δy
4	1	0.82843
2	0.5	0.44949
1	0.25	0.23607
0.5	0.125	0.12132
0.1	0.025	0.02485

Table 1. *Linear approximation in action: (i)* $f(x) = \sqrt{x}$ *and* $x_0 = 4$.

A couple of things may be observed. First, the estimates become more accurate as Δx gets smaller. When $\Delta x = 4$, the estimate is off by more than 0.1, and when $\Delta x = 0.1$, the estimate is off by less than 0.0002. Second, all the estimates are *too big* [2] in this example.

Let's see what happens when we use the same function, and the same values of Δx, but with a different *starting point*, namely $x_0 = 25$. In this case the approximation formula gives

$$\Delta y = \sqrt{25 + \Delta x} - \sqrt{25} \approx \frac{1}{2} \cdot 25^{-1/2} \cdot \Delta x = \frac{\Delta x}{10},$$

and repeating the computations above produces the table

Δx	$f'(x_0) \cdot \Delta x$	Δy
4	0.4	0.38516
2	0.2	0.19615
1	0.1	0.09902
0.5	0.05	0.04975
0.1	0.01	0.00999

Table 2. *Linear approximation in action: (ii)* $f(x) = \sqrt{x}$ *and* $x_0 = 25$.

What do we see here? First of all, the same patterns we observed above still hold, namely better estimates for smaller values of Δx, and all the estimates are bigger than the actual differences. But we can also compare the results in the first table to the results in the second table, and we see that for the same values of Δx, the estimates in the second table ($x_0 = 25$) are much better[3] than the corresponding estimates in the first table ($x_0 = 4$).

This example illustrates two of the dependencies that I mentioned before, namely that the accuracy of the estimate provided by the linear approximation depends on the point (x_0) and on the size of Δx. The next example will show that there is also a strong dependence on the the *function*, $f(x)$.

Example 2. In this example, I'll apply use linear approximation with the function $f(x) = x^3$, and I'll generate the same table that I did in Example 0.0.1 for $x_0 = 4$, with the same values of Δx that I used before. I'll leave it to you, as an exercise, to produce the table for $x_0 = 25$.

In this case, $f'(x) = 3x^2$, and for $x_0 = 4$, linear approximation gives

$$\Delta y = (4 + \Delta x)^3 - 4^3 \approx f'(x_0)\Delta x = 3 \cdot 4^2 \cdot \Delta x = 48 \cdot \Delta x,$$

[2] Consider the geometric interpretation of the approximation formula, to understand why.

[3] The estimates in the $x_0 = 25$ table are about 10 times closer to the actual values than the corresponding estimates in the $x_0 = 4$ table.

and we obtain the table

Δx	$f'(x_0) \cdot \Delta x$	Δy
4	192	448
2	96	152
1	48	61
0.5	24	27.125
0.1	4.8	4.921

Linear approximation in action: (iii) $f(x) = x^3$ *and* $x_0 = 4$.

What do we observe here? The first thing to notice that the approximations are not nearly as good in this case as they were in the first example, ($f(x) = \sqrt{x}$, $x_0 = 4$). Not until $\Delta x = 0.1$ is the distance between the estimate, $48 \cdot \Delta x$, and the actual value, $(4+\Delta x)^3 - 4^3$, less than 1. This illustrates the third dependency that I mentioned before. Namely, *the accuracy of the approximation depends on the* **function** *too.* Linear approximation will yield very good approximations for this function, but we need smaller values of Δx to get them. For example, if $\Delta x = 0.01$ then

$$\Delta y = (4.01^3 - 4^3) = 0.481201 \quad \text{and} \quad 48 \cdot (0.01) = 0.48,$$

and the difference between the estimate and the actual value of Δy is about 0.0012.

There are other differences between this example and the first two above. For one thing, the estimates in this case are all *too small*, (see the footnote on the previous page). And, finally, as you should check by repeating the second half of Example 0.0.1 for yourself, if we increase x_0, then the estimated values of Δy will be worse for the same choices of Δx. (In Example 0.0.1, the accuracy of the estimates improved when x_0 increased.)

3. *Marginal functions and approximation in economics*

Economic activity is often described in terms of the *change* in the values of the economic functions being considered. Economists use the word '*marginal*' to describe this change. For example, the marginal revenue of a firm is defined to be the change in revenue generated by an increase in output of one unit. I.e., if $r = f(q)$ is the revenue function, where q is the firm's output and r is its revenue, then the marginal revenue is

$$mr = f(q+1) - f(q). \tag{1}$$

If the function in question is differentiable, then the approximation approximation formula can be used to estimate marginal behavior. For example, applying the approximation formula, (3), to Equation (1), above, we find that

$$mr = f(q+1) - f(q) \approx f'(q) \cdot \Delta q = f'(q) \cdot 1 = \frac{dr}{dq}. \tag{2}$$

In other words, the marginal revenue is approximately equal[4] to the derivative of the revenue function. All of this leads to the following definition.

> If the revenue function $r = f(q)$ is differentiable then the **marginal revenue function** is **defined** to be the **derivative** of the revenue function, dr/dq.

This definition is only one example. In general, in the context of differential calculus applied to economics, then the word *marginal* connotes *derivative*. Some of the most common examples are listed below.

[4] Strictly speaking, this statement is only accurate if the revenue function is '*well behaved*', since in general, the linear approximation formula is only accurate when Δq is 'sufficiently small', and for many functions, $\Delta q = 1$ is **not** sufficiently small.

- The **marginal cost** function is the derivative of the cost function, with respect to output.
- The **marginal propensity to consume** is the derivative of the consumption function with respect to income.
- The **marginal propensity to save** is the derivative of the savings function with respect to income.
- The **marginal revenue product** is the derivative of revenue with respect to labor input (number of employees).
- The **marginal product** of capital (or labor, or x, etc.) is the derivative of output with respect to capital (or labor, or x, etc.).

Example 3. Suppose that a firm's production function is given by

$$q = 100(m+4)^{2/3},$$

where q is the firm's output, and m is the number of the firm's employees. The firm's *marginal product of labor*[5] is

$$\frac{dq}{dm} = \frac{200}{3}(m+4)^{-1/3}.$$

Suppose that the firm's *current* workforce is $m_0 = 60$. By how much can the firm expect output to increase if they hire one more employee? According to the approximation formula

$$\Delta q \approx \left(\frac{dq}{dm}\bigg|_{m=60} \right) \cdot \Delta m = \frac{200}{3} \cdot 64^{-1/3} \cdot 1 = \frac{50}{3} \approx 16.667.$$

In other words, if the number of employees increases from 60 to 61, then the output will approximately increase by the value of the *marginal product function* when $m = 60$. How good is this approximation? Well,

$$q(61) - q(60) = 100 \cdot 65^{2/3} - 100 \cdot 64^{2/3} = 16.62356,$$

rounded to 5 decimal places, so the approximation in this case is reasonably good — the difference between the estimate and the actual change in output is less than 0.05.

Example 4. The consumption function for a small, developing country is estimated to be

$$C = \frac{9Y^2 + 4Y + 2}{10Y + 3},$$

where C is *per-capita* consumption and Y is *per-capita* income. Both C and Y are measured in 1000's of dollars. The current per-capita income is $Y_0 = 1.2$. What will the approximate change in consumption be if per-capita income increases by \$250?

The marginal propensity to consume[6] in this case is

$$\frac{dC}{dY} = \frac{(18Y+4)(10Y+3) - 10(9Y^2+4Y+2)}{(10Y+3)^2} = \frac{90Y^2 + 54Y - 8}{100Y^2 + 60Y + 9}.$$

The (projected) change in income is $\Delta Y = 0.25 = \frac{250}{1000}$, since Y is measured in 1000's of dollars, so the (projected) change in per-capita consumption is

$$\Delta C \approx \left(\frac{dC}{dY}\bigg|_{Y=1.2} \right) \cdot \Delta Y = 0.828\overline{444} \cdot 0.25 = 0.207\overline{111},$$

according to the approximation formula. In dollar terms, the projected per-capita increase in consumption is about \$207.

[5] The derivative in this example is computed using the *chain rule*.

[6] I used the quotient rule here.

There is a natural question to ask here. Namely, why use linear approximation (or any other type of approximation) to estimate the change in the firm's output? Why not compute the change directly? In the age of calculators (to say nothing of 'smart' phones), why compute an approximate value when the *precise* value is a few key-strokes away?

There are several reasons. First, for functions like $f(x) = x^{2/3}$, $s = e^t$ or $U = \ln v$, to name a few simple examples, the values that your calculator (or laptop) produces are usually approximations themselves, albeit very good ones. In other words, there is nothing wrong with using an approximation — much of (applied) mathematics involves finding good approximations. Linear approximation is one of the most basic (and important) approximation tools in the mathematical toolkit.

Second, from a practical point of view, the functions that economists use to model economic reality are produced using economic theory, sophisticated statistical and mathematical tools, ***from actual data***. But this data has gaps, and the approximation formula and its more sophisticated relatives can be used to fill these gaps, and in certain cases *predict* future values.

Finally, there are situations where the derivative of a function is known at a certain point but the values of the function itself at nearby points may be (somewhat) difficult to calculate.[7] In this case, linear approximation provides an easy way to gauge the change in the value of the function.

Exercises.

1. Find the equation of the tangent line to the graph $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ at the point $(1, 0)$.

2. The demand equation for a firm's product is $p = 200 - 0.3q^{2/3}$, where p is the price, measured in \$100s, and q is output (=demand) measured in 1000s of units.

(a) Find the firm's revenue function and the firm's marginal revenue function. What is the firm's marginal revenue when $q = 1000$?

(b) Use your answer to part a. to **estimate** the change in the firm's revenue when output increases from 1,000,000 units to 1,000,150.

Hints/Suggestions:

- Pay attention to the units for p, q and r, (r and p are measured in the same units).
- Use *linear approximation.*

3. The demand function for a monopolistic firm is $p = 300 - 0.4q$, and the firm's cost function is $c = 0.05q^2 + 30q + 1000$. Find the level of output, q, for which marginal revenue equals marginal cost.

4. The national savings function for Slugsylvania is

$$S = \frac{Y^2 - 13}{10Y + 77},$$

where Y is annual income and S is annual savings, and both are measured in \$ billions.

(a) Compute the marginal propensity to save and the marginal propensity to consume when $Y = 10$. Round your answer to the nearest million.

(b) Compute $\lim\limits_{Y \to \infty} \dfrac{dS}{dY}$ and interpret your answer in economic terms.

5. The production function for ACME Widgets is given by

$$q = (5l + 4)^{2/3},$$

[7] We will see this in 11B.

where q is the firm's weekly output, measured in 1000s of widgets and l is the firm's weekly labor input, measured in \$1000's. The firm's *marginal* revenue function is given by

$$\frac{dr}{dq} = 80q^{-1/4},$$

where revenue and marginal revenue are measured in \$1000's.

(a) Compute the firm's *marginal product of labor* and *marginal revenue product* when the firm's labor input is \$12,000 a week.

(b) By approximately how many widgets/week will the firm's output change if they increase the weekly labor input from \$12,000 to \$12,500?

(c) By approximately how much will the firm's revenue change if they increase the weekly labor input from \$12,000 to \$12,500?

6. The demand equation for a firm's product is $p = 250 - 0.2q$, where p is the price of the firm's product measured in dollars and q is the firm's weekly output. The firm's production function is $q = 20(10m+5)^{2/3}$, where m is the number of the firm's employees.

(a) What is the firm's output when $m = 12$?

(b) What are the firm's revenue and marginal revenue when $m = 12$?

(c) What is the firm's marginal physical product, (marginal product of labor), when $m = 12$?

(d) The monthly cost to the firm of hiring a 13th employee is \$4500. Should the firm hire the 13th employee? Explain your answer.