Exponential and logarithm functions

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The material in this supplement is assumed to be mostly *review* material. If you have never studied exponential and/or logarithm functions before then you should find a text that covers this material in greater detail (e.g., the full version of our textbook).

1. *Exponential functions*

***Definition* 1.**

A function of the form

$$y = b^x,$$

where x is the independent variable and b is a constant, is called an ***exponential function.***

Comments:

- The constant b that appears in the definition above is called the base of the exponential function.
- In order that the exponential function be defined for all (real) x the base is assumed to satisfy $b > 0$. We also assume that $b \neq 1$ so that $y = b^x$ is not a constant function.

***Fact* 1.**

The function $y = b^x$ is defined for all x and

$$b^x > 0 \tag{1.1}$$

for all x.

Exponential functions have very useful algebraic properties[†]:

***Fact* 2.**

a. $b^0 = 1$
b. $b^1 = b$
c. $b^{x_1+x_2} = b^{x_1} \cdot b^{x_2}$
d. $b^{-x} = 1/b^x$
e. $b^{ax} = (b^a)^x = (b^x)^a$

As mentioned in Fact 1, exponential functions are defined for all x. The graph of $y = b^x$ has one of two characteristic forms, depending on whether $0 < b < 1$ or $1 < b$.

[†]The list includes some redundancy, but these are the algebraic properties that we use the most, so I listed them all.

***Fact* 3.**

If $b > 1$ then the function $y = b^x$ increases (very rapidly) as the variable x increases, and approaches 0 when x takes very large negative values. If $0 < b < 1$ then $y = b^x$ approaches 0 as x increases, and grows very large when x takes large negative values.

Exponential functions grow very rapidly (when the base is greater than 1). People often use the expression '***growing exponentially***' to mean growing rapidly. To get a sense of how fast exponential growth really is, read on.

***Fact* 4.**

If $b > 1$ then the function $y = b^x$ grows (eventually) more rapidly than any power of the variable x. In other words, for any power k, no matter how large

$$b^x > x^k,$$

once x is large enough.

To get a better idea of what this is saying look at a numerical example. If $b = 1.1$ and $k = 100$, then when x is relatively small x^{100} will be bigger than 1.1^x, as the first few entries in the table below indicate. However when x increases, 1.1^x grows larger than x^{100}, and the difference, $1.1^x - x^{100}$, grows exponentially with x, (remember, 10^k is a 1 followed by k zeros).

x	x^{100}	1.1^x	$1.1^x - x^{100}$
2	1.26×10^{30}	1.21	-1.26×10^{30}
10	10^{100}	2.5937	-10^{100}
100	10^{200}	13780	-10^{200}
1000	10^{300}	2.47×10^{41}	-10^{300}
10000	10^{400}	8.449×10^{413}	8.449×10^{413}
100000	10^{500}	1.855×10^{4139}	1.855×10^{4139}

1.1 *Compound interest*

Exponential functions provide one of the simplest and most important models for growth, whether its growth of a population in the biological setting or growth in the value of an investment.

An investment with interest rate r, compounded annually, grows according to the simple rule that at the end of each year the amount $r \cdot S$ is added to the investment, where S is the value of the investment at the end of the previous year. The initial amount invested, P_0, is called the principal. We use the variable t to measure years, and let $S(t)$ denote the value of the investment after t years, then

$$\begin{aligned}
S(0) &= P_0 \\
S(1) &= P_0 + r \cdot P_0 = P_0(1+r) \\
S(2) &= P_0(1+r) + r \cdot P_0(1+r) = P_0(1+r)^2 \\
S(3) &= P_0(1+r)^2 + r \cdot P_0(1+r)^2 = P_0(1+r)^3 \\
&\vdots \\
S(t) &= P_0(1+r)^{t-1} + r \cdot P_0(1+r)^{t-1} = P_0(1+r)^t
\end{aligned}$$

Example 1.

Suppose that an investment earns the interest rate $r = 3\% \, (= 0.03)$, compounded annually, and the principal is $P_0 = \$1000.00$. After 1 year the value of the account is

$$S(1) = 1000(1 + 0.03) = \$1030.00.$$

After 5 years, if the money is left untouched, the value of the account will be

$$S(5) = 1000(1 + 0.03)^5 \approx 1000 \cdot 1.15927 = \$1159.27,$$

and after 10 years the account will be worth

$$S(10) = 1000(1 + 0.03)^{10} \approx 1000 \cdot 1.34392 = \$1343.92.$$

Different investments have different periods of compounding. The interest on some investments is compounded quarterly (4 times a year), on others it is compounded monthly (12 times a year), and frequently (e.g., standard savings accounts) the interest is compounded daily (365 times a year, or 360, depending on who you ask).

If an investment earns the annual interest rate r, compounded k times a year, then the year is divided into k periods, and at the end of each period the amount $(r/k) \cdot S$ is added to the value of the investment, where S is the value of the investment at the end of the previous period. We want to derive a formula for

$$S_k(t) = \text{ the value of the investment after } t \text{ years},$$

as we did for interest compounded annually. I'll do this in two steps (really just 1.2 steps).

First denote by $S(m)$ the value of the investment after m *periods* (days, months, etc.). Then, analogously to the case of interest compounded annually,

$$S(m) = P_0 \left(1 + \frac{r}{k}\right)^m, \tag{1.2}$$

where P_0 is the principal as before. Second, in t years there are $m = kt$ periods (k periods in each year), so formula (1.2) gives the following

***Fact* 5.**

If an investment earns an annual interest rate r, compounded k times a year, then the value of the investment after t years is given by

$$S_k(t) = P_0 \left(1 + \frac{r}{k}\right)^{kt}, \tag{1.3}$$

where P_0 is the initial investment (the principal*).*

Comments:

- The formula (1.3) includes the case of interest compounded annually, which corresponds to $k = 1$.
- The function $S_k(t)$ is an exponential function of t (multiplied by the constant P_0). The base of this exponential function is

  $$b_{k,r} = \left(1 + \frac{r}{k}\right)^k,$$

  since

  $$\left(1 + \frac{r}{k}\right)^{kt} = \left[\left(1 + \frac{r}{k}\right)^k\right]^t = (b_{k,r})^t.$$

Example 2. Suppose that the interest rate $r = 0.03$ from Example 1 is compounded daily (365 times). After 1 year, an initial investment of \$1000.00 will be worth

$$S_{365}(1) = 1000(1 + (0.03/365))^{365} \approx \$1030.45,$$

after 5 years the investment will be worth

$$S_{365}(5) = 1000(1 + (0.03/365))^{365 \cdot 5} \approx \$1161.83,$$

and after 10 years it will be worth

$$S_{365}(10) = 1000(1 + (0.03/365))^{365 \cdot 10} \approx \$1349.84.$$

Comparing the two examples, you can see that compounding interest more frequently increases the return on the investment. To see this more clearly, we can compute the *effective rate* of the investment.

Definition 2.

Given an interest rate r that is compounded k times a year, the effective rate*, $\tilde{r}$, is the interest rate that when compounded annually, earns the same as r compounded k times a year. In this context, the original interest rate r is called the* nominal rate*. The effective rate is computed using the formula*

$$\tilde{r} = \left(1 + \frac{r}{k}\right)^k - 1.$$

Example 3. The effective rate of $r = 3\%$, compounded 365 times a year is

$$\tilde{r} = \left(1 + \frac{0.03}{365}\right)^{365} - 1 \approx 3.045\%.$$

1.2 *'The' exponential function*

Among all possible bases for exponential functions there are three that are more commonly used than others. Base 10 is popular because most modern cultures count base 10, (because humans generally have 10 fingers). Computer scientists use base 2 in their computations because of the binary nature of computer architecture. However, from a calculus perspective the 'best' base for an exponential function is the number

$$e = 2.7182818284590452354\ldots$$

e is an irrational number, and it is the 'natural' base for an exponential function. You will understand this statement better the more calculus you know. Indeed, many people refer to the function

$$f(x) = e^x$$

as *the* exponential function, or the natural exponential function.

2. *Logarithms*

Suppose that \$1000.00 is invested in an account earning 5%, compounded annually. How many years will it take for the investment to double in value? To answer this question, we need to solve the equation

$$1000(1.05)^t = 2000,$$

for the variable t. Dividing both sides of this equation by 1000 reduces the problem to its basic form

$$(1.05)^t = 2. \tag{2.1}$$

In other words, t, is the exponent to which we have to raise 1.05 in order to get 2.

***Definition* 3.**

The ***logarithm*** *base b of x, is the number y satisfying*

$$b^y = x.$$

We write this as

$$y = \log_b x.$$

This is a function of x.

With the logarithm we can answer the question above. The solution, t_0, to the problem above is simply $t_0 = \log_{1.05} 2$.

From its definition, you can see that the logarithm function base b, $y = \log_b x$, is the ***inverse function*** of the exponential function $y = b^x$. Recall that $f(x)$ and $g(x)$ are inverse functions if

$$f(g(x)) = g(f(x)) = x.$$

You should check that $\log_b x$ and b^x satisfy this relation.

The domain of definition of log functions is also determined by its definition. Specifically, since $\log_b x = y$ implies that $x = b^y$, it follows that $x > 0$, since b^y is always strictly greater than 0. In other words

***Fact* 6.**

The function $y = \log_b x$ is defined for $x > 0$.

The algebraic properties of exponential functions (Fact 2) give rise to corresponding algebraic properties of log functions.

***Fact* 7.**

a. $\log_b 1 = 0$
b. $\log_b b = 1$
c. $\log_b(X \cdot Y) = \log_b X + \log_b Y$
d. $\log_b(1/X) = -\log_b X$
e. $\log_b(X^\alpha) = \alpha \cdot (\log_b X)$

***Example* 4.**

$$\log_5 \left[\frac{5x^2y^3}{z^4}\right] = 1 + 2\log_5 x + 3\log_5 y - 4\log_5 z.$$

Just as every exponential function can be expressed in terms of any other exponential function, every log function can be expressed in terms of any other log function. This simple, but important fact is summarized in the *change of base formula*

***Fact* 8.**

$$\log_b x = \frac{\log_a x}{\log_a b},$$

for any bases a and b. This is called the ***change of base formula****.*

Proof: Suppose that $\log_b x = L$, so that $b^L = x$. Use property e. from Fact 7, to see that

$$\log_a x = \log_a(b^L) = L \cdot (\log_a b).$$

Now divide both sides above by $\log_a b$ to get the change of base formula

$$\log_b x = L = \frac{\log_a x}{\log_a b}.$$

2.1 *The natural log function*

Just as the number e is the natural choice for the base of an exponential function, e is also the natural choice for the base of a log function. As I mentioned before, we need to know a bit more calculus to understand what 'natural' means in this context. The function $\log_e x$ is therefore called the *natural log function*[‡] (and sometimes just 'the' log function), and many people use the special notation[§]

$$\log_e x = \ln x.$$

***Fact* 9.**

Every log function is a constant multiple of the natural log function.

Proof: This follows directly from the change of base formula, Fact 8, with a replaced by e:

$$\log_b x = \frac{\ln x}{\ln b} = \left(\frac{1}{\ln b}\right) \cdot \ln x.$$

Remember: $(1/\ln b)$ is a constant (that depends on b).

Using the natural log function, we can also express every exponential function in terms of *the* exponential function (e^x) in a simple way.

***Fact* 10.**

For any base b,

$$b^x = e^{x \ln b}.$$

Proof: The natural log of b^x is $\ln(b^x) = x \ln b$, and this means that

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

Comment: These two facts will be useful when we learn to differentiate, since the derivatives of e^x and $\ln x$ are particularly simple.

***Exercise* 1.**

Use your calculator to compute $\log_7 11$ and $\log_3 10$.

[‡] $log_{10} x$ is often called the *common logarithm.*

[§] Though in many advanced math texts, people simply write $\log x$ when they mean $\log_e x$. This can be confusing, since in many lower division math texts, (and on calculators), $\log x$ is used to mean $\log_{10} x$.

***Exercise* 2.**

Use the functions e^x and $\ln x$ on your calculator to compute 3^2 and 2^3 (you can check your own work, since you know the answers to these two). Repeat this to compute $1.05^{3.14}$.

3. *A note on computation*

These days most of us rely on calculators of one form or another to compute the actual values of most basic functions. This is certainly the case for functions like e^x and $\ln x$. The question is: how does the calculator compute the values of these functions?

All of the computational operations that we perform, no matter how sophisticated are ultimately based on the *elementary* operations of addition and multiplication. Even computers function based on simple addition-like operations. After the elementary operations, the next level of complexity in computation is *extracting (integer) roots*, like square roots, cube roots, etc. To extract a root[¶] you need to solve a polynomial equation. E.g., computing $\sqrt[3]{7}$ is equivalent to solving the equation $x^3 = 7$ for the variable x.

Functions that can be defined using elementary operations and extracting roots are called *algebraic functions*.

Example 5. Polynomials and rational functions (quotients of polynomials) are simple algebraic functions. E.g.,

$$P(x) = 3x^4 - 2x^2 + 7x - 11 \quad \text{and} \quad R(x) = \frac{x^2 + 3x + 1}{3x^3 - 5x^2 + 2x - 1}$$

are both defined in terms of multiplication, addition, subtraction and division and nothing else. The function

$$\sqrt[3]{x^2 + x + 2}$$

is a slightly more involved algebraic function.

Functions that cannot be built up in a *finite* number of steps from the elementary operations and root extractions are called *transcendental functions*.[‖] Exponential and logarithmic functions fall into this category. To compute the precise value of e^x or $\ln x$, we need to evaluate the *mathematical limit* of certain polynomials whose degree is getting infinitely large.

For example

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots,$$

where $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ is the *factorial* function. And, if $|x| < 1$ then

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

In both the equations above, the final ellipses ($\cdots$) indicate that we need to continue to add (or subtract) the appropriate terms forever. This is impossible from a practical point of view. On the other hand, it is possible to compute very good approximations for e^x if we compute the finite polynomial

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!},$$

[¶]Without using the exponential and natural log functions.

[‖]Because they *transcend* algebraic operations.

for a very large value of n, and similarly for $\ln(1+x)$.

Electronic calculators and computers use these ideas as well as large tables of values that have been computed and some clever algorithms to compute e^x and $\ln x$. The values that they produce are approximations, but very good ones.

***Exercise* 3.**

Use your calculator to compute e^2. Remember, the number that pops up is also an approximation, but it's a decent one, and we'll take it as the 'correct' value. Now compute the value of the polynomial

$$E_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

for $x = 2$ and larger and larger values of n. How big does n have to be before the first 2 decimal digits of $E_n(2)$ agree with the 'correct' value e^2 that your calculator produced?

***Exercise* 4.**

Repeat the previous exercise to compute an approximation to $\ln 2$ using the polynomial

$$L_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n}$$

to approximate $\ln(1+x)$. Note that $L_n(x)$ gives accurate approximations of $\ln(1+x)$ only when $-1 < x < 1$. On the other hand $\ln 2 = -\ln 0.5$, and $0.5 = 1 + (-0.5)$. You do the rest. How big does n have to be before your approximation agrees with the first two decimal digits of your calculator's answer?