Modelling Economic Variables

1. Mathematical models

The two central topics of AMS/Econ 11A are differential calculus on the one hand, and its applications to business and economics on the other. Economists use mathematics to model the behavior of a wide variety of economic variables and their interactions. These models can be very simple, describing nothing more than a linear or quadratic relationship between two variables, and they can be complex, involving many variables and intricate functional relationships. Economists use purely mathematical tools like calculus and linear algebra to analyze these models and better understand the hypothetical relationships between the economic variables.

Most mathematical models of economic behavior boil down to an equation or, more generally, a system of equations. Broadly speaking, these equations describe three distinct types of relationships.

- *Identities* which serve as definitions. For example, the equation
$$\pi = r - c,$$
identifies profit (π) as the difference between revenue (r) and cost (c). This is the definition of profit.
- *Functional* equations which describe how one variable depends on others. For example the equation
$$Q = AK^{\alpha}L^{1-\alpha}$$
describes how a firm's output (Q) depends on capital and labor input (K and L, respectively).
- *Equilibrium* equations. These are equations, or systems of equations, that describe structural equilibria in various economic systems. An equilibrium point is literally a point of balance and in economics, equilibrium refers to a state wherein all of the various market forces are in balance and the system as a whole is stable, or unmoving. The best known equilibrium equation in economics is
$$Q_s = Q_d,$$
i.e., quantity supplied = quantity demanded (where the variable Q will always signify quantity, in some form or another). The solution of this equation is the point of *market equilibrium*, which is the point where a market stabilizes and determines the price of a good and the quantity of that good that is trading hands.

In the sections that follow I will survey some of the basic economic variables that we will encounter in the course, and some of the relations between them.

2. *Price and Quantity*

The variables p and q (lower case or upper case) will always signify the price and quantity variables. These are perhaps the two most basic variables in economics, and they appear in many of the examples in this course. Price is always measured in some form of currency, e.g., dollars, \$1000's, etc. Quantity can be measured in many ways, e.g., single units, 100s of units, gallons, etc.

2.1 *Supply and Demand*

A *demand* equation, (or demand function), describes the relationship between the price of a product, p, and the quantity demanded of this product, q. A demand equation is always characterized by the fact that both p and q represent positive quantities, and that p is a *decreasing* function of q and vice versa. This means that the graph of a demand equation is downward sloping.

Example 1. The equation

$$p + 0.05q = 100,$$

is a valid demand equation, as long as both p and q are positive.

If we rearrange the equation by moving the q-term to the right hand side we obtain an equivalent equation that has the advantage of expressing p as a function of q:

$$p = 100 - 0.05q. \tag{2.1}$$

Example 2. A *supply* equation (or function) describes the relationship between the price of a commodity and the amount that a producer of the commodity is willing to supply. Once again the variables should both be positive, but in this case, the higher the price, the more the producers will want to supply, and so we expect p and q to increase together.

For example the function

$$p = \sqrt{100 + q} \tag{2.2}$$

is a valid supply equation.

Example 3. Using the demand and supply equations of the previous two examples, we can find the point of *market equilibrium* by setting the two prices (or the two quantities) equal to each other. I.e., when the market is in equilibrium, the price will be such that the quantity supplied and the quantity demanded at that price are the same,

$$\sqrt{100 + q} = 100 - 0.05q. \tag{2.3}$$

To find the equilibrium quantity, we square both sides of the equation, obtaining

$$100 + q = (100 - 0.05q)^2 = 10000 - 10q + 0.0025q^2,$$

or

$$0.0025q^2 - 11q + 9900 = 0.$$

This is a quadratic equation which we solve using the quadratic formula[†] giving two possible values for q: $q_0 \approx 1261.92$ and $q_1 \approx 3138.08$. Of these two solutions to the quadratic equation

[†]See SN 2 if the quadratic formula seems mysterious.

above, only q_0 is a solution of the original equation, (2.3).[‡] Thus the equilibrium quantity is q_0 and the corresponding equilibrium price is $p_0 = 100 - 0.05q_0 \approx 36.904$.

3. *Revenue, Cost and Profit*

3.1 *Revenue*

The revenue is the total amount of money earned by a firm in a given unit of time. Revenue is typically denoted by the variable r. In basic examples, the revenue of a firm is given by the quantity of the product that they sell multiplied by the price per unit of that product. In other words

$$r = p \cdot q,$$

where p and q are the price and quantity variables described above. When the price is determined by the quantity alone, e.g., by the firm's demand equation,[§] we can also describe r as a function of q alone.

Example 4. We can use the demand function defined in Example 1 to obtain a revenue function:

$$r = p \cdot q = (100 - 0.05q) \cdot q = 100q - 0.05q^2. \tag{3.1}$$

The revenue variable is typically measured in the same units as the price variable.

3.2 *Cost*

A firm's total cost is represented by the variable c. Total cost can be split into *fixed cost* (FC) and *variable cost* (VC). The fixed cost is the amount of money spent by the firm independently of the quantity produced, even if nothing is produced, due to things like rent. Variable cost is the part of the total cost that depends directly on the quantity produced.

Example 5. Suppose that a small firm has fixed monthly costs (rent, utilities, etc.) of \$2000 and that its variable costs are \$10 per unit, then the firm's total (monthly) cost is given by function

$$c = 10q + 2000. \tag{3.2}$$

3.3 *Profit*

A firm's profit is the difference between its revenue and its cost. Profit is often denoted by the Greek letter π, (to distinguish it from price p), and with r being revenue and c being cost, we have

$$\pi = r - c.$$

Example 6. Suppose that the revenue function in Example 4 represents a firm's monthly revenue in dollars based on monthly sales of q units of its product, and suppose that the cost function in Example 5 represents its monthly costs. Then the firm's profit function is given by

$$\pi = r - c = (100q - 0.05q^2) - (10q + 2000) = -0.05q^2 + 90q - 2000. \tag{3.3}$$

[‡] Squaring both sides of an equation can often lead to ***extraneous*** solutions, as in the example here. Notice that plugging q_1 into the demand equation above leads to a ***negative*** price, which contradicts the assumptions of the model.

[§] This assumption works for monopolistic firms, so called *price-setters*.

4. *Consumption, Savings and Income*

The (macroeconomic) relationship between income, consumption and savings is very important to economists. Income is often represented by the variable Y, (though our textbook uses I), consumption by the variable C and savings by S, (note that all of these variables are typically upper case).

The fundamental relationship between consumption, savings and income is given by the elementary *accounting* identity

$$Y = C + S, \tag{4.1}$$

in other words, all of the income is either consumed or saved.

Example 7. Suppose that the consumption function for a developing country is given by

$$C = \frac{9Y^2 + Y}{10Y + 3}, \tag{4.2}$$

where C and Y are both measured in millions of dollars a year. What is the savings function?

Because of Equation (4.1) this question is easy to answer. Namely, $S = Y - C$, so here

$$S = Y - \frac{9Y^2 + Y}{10Y + 3} = \frac{Y^2 + 2Y}{10Y + 3}.$$

5. *Input and output*

Closely related to supply and cost functions are output, or production functions. The production function of a firm, (or more generally, of an industry or an economy), is the function that describes the firm's output as it depends on the various inputs it uses in the production process. In basic models of production, output is described as depending on two inputs, *capital input* and *labor input*, i.e.,

$$Q = F(K, L).$$

Often, all three variables are measured in units of currency, so that the function is actually describing how the value of the output depends on values of the inputs.

Example 8. A popular model of production is given by the Cobb-Douglas function,[¶]

$$Q = AK^{\alpha}L^{\beta},$$

where Q is output, and K and L are the capital and labor inputs, respectively.

In this model it is assumed that the other three parameters, A, α and β are positive constants, with the further restriction that

$$0 < \alpha < 1 \text{ and } 0 < \beta < 1.$$

Example 9. A firm's monthly output q is given by

$$q = 75\sqrt{m - 5},$$

[¶]Strictly speaking, in a classical Cobb-Douglas function, $\alpha + \beta = 1$, which guarantees constant returns to scale, among other things.

where m is the number of the firm's (full-time) employees. A function like this, which depends only on labor input (m in this case) is sometimes used to model *short term* production. It is assumed that the capital input is fixed in the short term, and only the labor input varies.

Example 10. Suppose that the demand equation for the firm from Example 9, above, is given by

$$p = \frac{2000}{1 + 2\sqrt{q}},$$

where p is the price per unit of the firm's product and q is the (monthly) demand for the firm's product. Find the firm's monthly revenue if it currently has $m = 14$ employees, assuming that it sells every unit it produces.

Based on the production function in Example 9, if the firm has 14 full time employees, then its monthly output is

$$q(14) = 75\sqrt{9} = 225.$$

Since the firm sells every unit it produces, the (current) price of the firm's product is

$$p = \frac{2000}{1 + \sqrt{225}} = \frac{2000}{16} = 125,$$

so the firm's monthly revenue is

$$r = p \cdot q = 125 \cdot 225 = 28125.$$