Solutions

1. The consumption function for a small country is given by

$$C = \ln\left(\frac{e^{0.95Y}}{e^{0.2Y}+5}\right),$$

where Y is national income, measured in \$ billions.

a. When $Y = 10$ consumption is ...

$C(10) = \ln\left(\frac{e^{9.5}}{e^2+5}\right) \approx 6.983$, (about 6.983 billion dollars).

b. First note that $C = \ln e^{0.95Y} - \ln\left(e^{0.2Y}+5\right) = 0.95Y - \ln\left(e^{0.2Y}+5\right)$.

So $\dfrac{dC}{dY} = 0.95 - \dfrac{0.2e^{0.2Y}}{e^{0.2Y}+5}$, and $\left.\dfrac{dC}{dY}\right|_{Y=10} = 0.95 - \dfrac{0.2e^2}{e^2+5} \approx 0.831.$

c. By approximately how much will *consumption and savings* increase if income increases from \$10 billion to \$10.4 billion?

First consumption, using linear approximation:

$$\Delta C \approx \left.\frac{dC}{dY}\right|_{Y=10} \cdot \Delta Y \approx 0.831 \cdot 0.4 = 0.3324$$

I.e., if income increases from \$10 billion to \$10.4 billion, consumption will increase by about \$332,400,000.

Next, savings, Recall that $\dfrac{dS}{dY} = 1 - \dfrac{dC}{dY}$, so $\left.\dfrac{dS}{dY}\right|_{Y=10} \approx 1 - 0.831 = 0.169$. Therefore, by the linear approximation again

$$\Delta S \approx \left.\frac{dS}{dY}\right|_{Y=10} \cdot \Delta Y \implies \Delta S \approx 0.169 \cdot 0.4 = 0.0676.$$

Alternatively, it is also true that $\Delta Y = \Delta S + \Delta C$, so

$$\Delta S = \Delta Y - \Delta C \approx 0.4 - 0.3324 = 0.0676.$$

Either way, if income increases from \$10 billion to \$10.4 billion, savings will increase by about \$67,600,000.00

d. $$\begin{aligned}\lim_{Y\to\infty}\frac{dC}{dY} &= \lim_{Y\to\infty}\left(0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)\\ &= \lim_{Y\to\infty} 0.95 - \lim_{Y\to\infty}\left(\frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)\\ &= 0.95 - \lim_{Y\to\infty}\left(\frac{0.2}{1+5e^{-0.2Y}}\right) \quad \text{(Divide numerator and denominator by } e^{0.2Y}.\text{)}\end{aligned}$$

$$= 0.95 - \frac{0.2}{1} = 0.75. \qquad\qquad (\text{Since } \lim_{Y\to\infty} e^{-0.2Y} = 0.)$$

Interpretation: When income grows sufficiently large, the MPC will stabilize at about 0.75, i.e., when income is large, the nation will consume about 75 cents of each additional dollar of income (and save the other 25 cents).

2. A firm's production function is $q = 30(4m - 16)^{1/3}$, where q is measured in 1000s of units and m is the firm's labor input measured in 40-hour work weeks (e.g., if $m = 5$, then the firm's employees are working a combined 200 hours a week and if $m = 17.2$, then the firm's employees are working a combined 4288 hours a week). Find the *labor-elasticity of output* for this firm when $m = 20$. Use your answer to estimate the *percentage* change in output, if the firm increases its labor input by 30 hours a week.

The labor-elasticity of output is given by

$$\eta_{q/m} = \frac{dq}{dm} \cdot \frac{m}{q}.$$

When $m = 20$, we find that $q = 30 \cdot 64^{1/3} = 120$ and

$$\left.\frac{dq}{dm}\right|_{m=20} = 30 \cdot \frac{1}{3} \cdot (4m-16)^{-2/3}\Big|_{m=20} \cdot 4 = \ 40(4m-16)^{-2/3}\Big|_{m=20} = \frac{40}{16} = 2.5,$$

so

$$\eta_{q/m}\Big|_{m=20} = 2.5 \cdot \frac{20}{120} = \frac{5}{12}.$$

To estimate the percentage change in output, we use the approximation

$$\%\Delta q \approx \eta_{q/m} \cdot \%\Delta m.$$

The percentage change in labor input is

$$\%\Delta m = \frac{\Delta m}{m} \cdot 100\% = \frac{0.75}{20} \cdot 100\% = 3.75\%$$

so the percentage change in output will be

$$\%\Delta q \approx \frac{5}{12} \cdot 3.75\% = 1.5625\%$$

Comment: If we simply evaluate the production function at $m_1 = 20.75$ we get

$$q(20.75) \approx 121.8464 \implies \%\Delta q = \frac{\Delta q}{q} \cdot 100\% \approx \frac{1.8464}{120} \cdot 100\% \approx 1.5387\%$$

so the approximation is reasonably accurate here.

3. The demand equation for a monopolist's product is $p = 250 - 0.2q$.

a. Find the price-elasticity of demand (as a function of q).

$$\eta_{q/p} = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{250-0.2q}{q}}{-0.2} = \frac{250-0.2q}{-0.2q} = \frac{q-1250}{q} = 1 - \frac{1250}{q}.$$

b. What is the price elasticity of demand when $p = \$50$? Is demand elastic, inelastic, or does demand have unit elasticity at this point?

To answer this, we first need to find the value of q when $p = 50$:

$$p = 50 \implies 50 = 250 - 0.2q \implies 0.2q = 200 \implies q = 1000.$$

Now, compute $\eta = 1 - \frac{1250}{1000} = -0.25$. Since $|\eta| < 1$, it follows that demand is *inelastic* at that point on the demand curve.

c. Suppose that the price is lowered (from \$50) to \$49.25. Use your answer to part b. to estimate the *percentage* change in demand.

We use the approximation, $\%\Delta q \approx \eta \cdot (\%\Delta p)$ and first we need to compute the percentage change in price:

$$\%\Delta p = \frac{p_{\text{new}} - p_{\text{old}}}{p_{\text{old}}} \cdot 100\% = \frac{49.25 - 50}{50} \cdot 100\% = -1.5\%.$$

Now, we can compute the (approximate) percentage change in demand,

$$\%\Delta q \approx (-0.25)(-1.5\%) = 0.375\%.$$

I.e., demand will *increase* by approximately $\frac{3}{8}$ of 1 percent.

d. What effect will this change in price have on the firm's revenue? Be as precise as you can, and explain your answer.

First, we refer to the relationship between elasticity and marginal revenue:

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right).$$

In the example above, when $q = 1000$ and $p = 50$, $\eta = -0.25$, so

$$\frac{dr}{dq} = 50(1 - 1/0.25) = 50(1-4) = -150 < 0.$$

Since marginal revenue is negative when $q = 1000$, an increase in q brings about a *decrease* in revenue, and since lowering the price raises demand, the effect on revenue of lowering the price will be to lower revenue.

More precisely, $\%\Delta q \approx 0.375\%$ when $q_0 = 1000$ and $\Delta p = -0.75$, it follows that $\Delta q \approx 3.75$, so the change in revenue will be

$$\Delta r \approx \left.\frac{dr}{dq}\right|_{q=1000} \cdot \Delta q \approx -150 \cdot 3.75 = -562.50.$$