Solutions

1. (a) Find the derivative of $f(x) = \sqrt[3]{x}$ at the point $x = 8$.

$$f(x) = \sqrt[3]{x} = x^{1/3} \implies f'(x) = \frac{1}{3}x^{-2/3} \implies f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$$

(b) Use your answer to (a) and *linear approximation* to estimate $\sqrt[3]{9}$.

Linear approximation: *if x_1 is close to x_0,*[†] *then*

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0).$$

In this case $x_0 = 8$ and $x_1 = 9$, so

$$\sqrt[3]{9} = f(9) \approx f(8) + f'(8)(9-8) = 2 + \frac{1}{12} \cdot 1 = \frac{25}{12} = 2.08333\ldots.$$

The calculator estimate is $\sqrt[3]{9} \approx 2.0800838\ldots$, so the linear approximation above is off by less than 0.0033.

(c) Use the same ideas to estimate $\sqrt{102}$.

Use linear approximation for the function $f(x) = \sqrt{x} = x^{1/2}$ and the points $x_0 = 100$ and $x_1 = 102$.[‡] With these choices, we have

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(100) = \frac{1}{20},$$

and therefore

$$\sqrt{102} = f(102) \approx f(100) + f'(100)(102 - 100) = 10 + \frac{1}{20} \cdot 2 = 10.1.$$

The calculator estimate is $\sqrt{100} \approx 10.0995049\ldots$, so the linear approximation above is off by less than 0.0005.

2. A firm's marginal revenue function is given by

$$\frac{dr}{dq} = 0.7q - 0.05q^2,$$

where revenue r is measured in \$1000s and output q is measured in 100s of units. By approximately how much will the firm's revenue change if output increases from 1000 units to 1050 units?

[†] There is no uniform definition for 'close'. You may assume that the points are close enough if you are asked to use linear approximation. There are methods for estimating the size of the error of approximation, depending on the size of $|x_1 - x_0|$, but we won't go into that at this point.

[‡] The idea is to find a point x_0 such that (a) x_0 is close to the point x_1 ($x_1 = 102$ in this case), and (b) $f(x)$ and $f'(x)$ are relatively easy to evaluate at x_0.

Note that we don't know the revenue function in this case, so we can't compute the revenue directly at any point,[§] nonetheless, we can estimate the *change* in the revenue using linear approximation. Specifically, we use linear approximation in this form:

$$r(q_1) - r(q_0) \approx \left(\left.\frac{dr}{dq}\right|_{q=q_0} \right)(q_1 - q_0).$$

In this problem, $q_0 = 10$ and $q_1 = 10.5$, because we are measuring output in 100s of units, and therefore

$$\left.\frac{dr}{dq}\right|_{q=10} = 7 - 5 = 2$$

so

$$r(10.5) - r(10) \approx 2 \cdot 0.5 = 1,$$

which means that the firm's revenue will increase by about \$1000.

3. Compute the derivatives of the functions below

a. $f(x) = \sqrt[3]{x^3 - 3x^2 + 1}$ **c.** $g(x) = \log_5 x$ **e.** $y = \ln(x^2 + 2x + 3)$

b. $s = e^{0.05t}$ **d.** $f(x) = 3x^2 e^x$ **f.** $k(u) = \dfrac{u \ln u}{2u + 1}$

The derivatives....

a. $f'(x) = \dfrac{1}{3}(x^3 - 3x^2 + 1)^{-2/3} \cdot (3x^2 - 6x) = \dfrac{x^2 - 2x}{(x^3 - 3x^2 + 1)^{2/3}}$. *power rule.*

b. $\dfrac{ds}{dt} = (0.05)e^{0.05t}$. *chain rule*

c. $g'(x) = \dfrac{1}{\ln 5} \cdot \dfrac{1}{x}$. $\log_5 x = \dfrac{\ln x}{\ln 5}$.

d. $f'(x) = 6xe^x + 3x^2 e^x = 3(2x + x^2)e^x$. *Product rule.*

e. $y' = \dfrac{2x + 2}{x^2 + 2x + 3}$. *chain rule.*

f. $k'(u) = \dfrac{(1 \cdot \ln u + u \cdot \frac{1}{u})(2u + 1) - 2(u \ln u)}{(2u + 1)^2}$ *Quotient rule* and *product rule*

$= \dfrac{(\ln u + 1)(2u + 1) - 2u \ln u}{(2u + 1)^2}$ for the numerator.

$= \dfrac{2u + \ln u + 1}{(2u + 1)^2}$ Then clean up.

4. The *marginal revenue* function of a monopolistic firm is given by

$$\frac{dr}{dq} = \sqrt{250 - q},$$

[§] We will learn how to do this in 11B.

where revenue is measured in \$1000s per month and the firm's output q is measured in 1000s of units per month. The firm's production function is

$$q = 30(4m - 15)^{1/3},$$

where m is the firm's labor input measured in 40-hour work weeks (e.g., if $m = 5$, then the firm's employees are working a combined 200 hours a week and if $m = 17.2$, then the firm's employees are working a combined 4288 hours a week). The firm's current labor input is $m = 35$.

a. Find the firm's output and *marginal product of labor* at the current level of labor input.

When labor input is $m = 35$, output is $q(35) = 30(4 \cdot 35 - 15)^{1/3} = 150$ and

$$\frac{dq}{dm} = 30 \cdot \frac{1}{3}(4m - 15)^{-2/3} \cdot 4 = 40(4m - 15)^{-2/3} \implies \left.\frac{dq}{dm}\right|_{m=35} = 40 \cdot 125^{-2/3} = 1.6.$$

b. Find the firm's *marginal revenue product* at the current level of labor input.

Marginal revenue product is dr/dm, and by the chain rule

$$\left.\frac{dr}{dm}\right|_{m=35} = \left.\frac{dr}{dq}\right|_{m=35} \cdot \left.\frac{dq}{dm}\right|_{m=35} = \left.\frac{dr}{dq}\right|_{q=150} \cdot \left.\frac{dq}{dm}\right|_{m=35} = \sqrt{100} \cdot 1.6 = 16.$$

c. The firm decides to increase its labor force and hires an additional part-time laborer, to work 10 hours a week. The total monthly expense to the firm for the new employee is \$2450.00. Use your answer to part **b.** to estimate the change in the firm's monthly profit.

If the firm increases its labor input by $\Delta m = \dfrac{10}{40} = 0.25$, then the change to the firm's revenue is

$$\Delta r \approx \left.\frac{dr}{dm}\right|_{m=35} \cdot \Delta m = 16 \cdot 0.25 = 4.$$

Revenue is measured in \$1000s, so the firm's revenue will increase by about \$4000.00, so its profit will increase by about \$4000.00-\$2450.00=\$1550.00.