Solutions

1. Find the absolute maximum and minimum values of function $f(x) = 2x^3 - 3x^2 - 12x + 11$ on the interval $[0, 10]$. Justify your claim that the values you found are indeed the max and min values.

This is a *closed interval* optimization problem: the maximum and minimum values occur at critical points of the function and/or at the endpoints of the interval. So, all we need to do here is find the critical point(s) of $f(x)$ in $[0, 10]$ and then evaluate $f(x)$ at these points and at the endpoints—the biggest value we see (on this finite list) is the maximum and the smallest value we see is the minimum.

Critical point(s):

$$f'(x) = 0 \implies 6x^2 - 6x - 12 = 0 \implies 6(x-2)(x+1) = 0 \implies \begin{cases} x_1 &= -1 \\ x_2 &= 2 \end{cases}$$

Only $x_2 = 2$ is in the interval $[0, 10]$, so we evaluate $f(x)$ at $x = 0$, $x = 2$ and $x = 10$:

$$f(0) = 11, \quad f(2) = -9 \leftarrow \boldsymbol{minimum} \quad \text{and} \quad f(10) = 1591 \leftarrow \boldsymbol{maximum}$$

2. Find the *absolute minimum* value of the function $c = 0.1q + 15 + \dfrac{100}{q}$ in the interval $(0, \infty)$. Explain how you know that the value you found is the absolute minimum.

First, critical point(s):

$$\frac{dc}{dq} = 0 \implies 0.1 - \frac{100}{q^2} = 0 \implies 0.1q^2 = 100 \implies q^2 = 1000 \implies q = \pm\sqrt{1000}$$

Of the two critical points, only $q^* = \sqrt{1000}$ (≈ 31.623) lies in the interval $(0, \infty)$. I.e., there is only one critical point in $(0, \infty)$.

Next, choosing a point between 0 and $\sqrt{1000}$, we have

$$\left.\frac{dc}{dq}\right|_{q=1} = -99.9 < 0$$

and choosing a point between $\sqrt{1000}$ and ∞, we have

$$\left.\frac{dc}{dq}\right|_{q=100} = 0.09 > 0.$$

Taken together, this means that c is decreasing in the interval $(0, \sqrt{1000})$ and increasing in the interval $(\sqrt{1000}, \infty)$, which implies that

$$c(\sqrt{1000}) \approx 21.325$$

is the absolute minimum value in $(0, \infty)$.

3. Consider the function $v = u^2 e^{-5u}$.

Before addressing the four questions below, I'll find the critical points of v:

$$\frac{dv}{du} = 2ue^{-5u} - 5u^2e^{-5u} = ue^{-5u}(2-5u)$$

so

$$\frac{dv}{du} = 0 \implies u = 0 \text{ or } u = \frac{2}{5}$$

a. Does v have an absolute *maximum* value in the interval $(0, \infty)$? If so, find it and justify your claim. If not, explain why not.

Yes. The point $u = 0$ is not in the *open* interval $(0, \infty)$, so there is only one critical point there: $u^* = 2/5$. Testing the first derivative on either side of u^*, we find that

$$\left.\frac{dv}{du}\right|_{u=0.1} = 0.1e^{-0.5} > 0 \text{ and } \left.\frac{dv}{du}\right|_{u=1} = -3e^{-5} < 0$$

so v is increasing from 0 to 2/5 and decreasing from 2/5 to ∞, and therefore

$$v(2/5) = \frac{4}{25}e^{-2} \approx 0.022$$

is the absolute maximum value in $(0, \infty)$.

b. Does v have an absolute *minimum* value in the interval $(0, \infty)$? If so, find it and justify your claim. If not, explain why not.

No. If $0 < u_1 \leq 2/5$, then $v(u_1) > v(u)$ for any $0 < u < u_1$, because v is *increasing* between 0 and 2/5. This means that $v(u_1)$ is *not* the absolute minimum value of v. Likewise, if $2/5 < u_2 < \infty$, then $v(u_2) > v(u)$ for any $u_2 < u < \infty$, because v is *decreasing* between 2/5 and ∞, and again, this means that $v(u_2)$ is *not* the absolute minimum value of v. In conclusion, there is no point u in $(0, \infty)$ such that $v(u)$ is minimum value.

c. Does v have an absolute *maximum* value in the interval $(-\infty, \infty)$? If so, find it and justify your claim. If not, explain why not.

No. First of all,

$$v(-1) = e^5 > \frac{4}{25}e^{-2} = v(2/5)$$

so $v(2/5)$ is not the absolute maximum value in $(-\infty, \infty)$. Additionally, since

$$v(2/5) \geq v(u)$$

for all $u \geq 0$ (see part a.), it follows that if $u \geq 0$, then $v(u)$ cannot be the absolute maximum value of v.

Next, if $u < 0$, then

$$\frac{dv}{du} = ue^{-5u}(2-5u) = ue^{-5u}(2+5|u|) < 0$$

(because $u < 0$, $e^{-5u} > 0$ and $(2 + 5|u|) > 0$) which means that v is *decreasing* in $(-\infty, 0)$. This means that given any $\tilde{u} < 0$, if $u < \tilde{u}$, then $v(u) > v(\tilde{u})$, so $v(\tilde{u})$ is *not* the maximum value of v.

It follows that there is no $\tilde{u}$ such that $v(\tilde{u}) \geq v(u)$ for all u in $(-\infty, \infty)$, so v has no absolute maximum in $(-\infty, \infty)$.

d. Does v have an absolute *minimum* value in the interval $(-\infty, \infty)$? If so, find it and justify your claim. If not, explain why not.

Yes. (No derivatives required)... $v(0) = 0$, but if $u \neq 0$, then

$$v(u) = u^2 e^{-5u} > 0 = v(0)$$

because $u^2 > 0$ and $e^{-5u} > 0$ and the product of positive numbers is positive. I.e., $v(0) = 0$ is the absolute minimum value of v in $(-\infty, \infty)$.