Solutions

1. A certificate of deposit offers an interest rate of 5.5% compounded daily for a term of 4 years. How much would you have to invest initially to have \$25000 when the CD matures?

If P_0 is invested at an interest rate r, compounded k times per year, then the value of the investment (the compound amount) after t years is given by

$$P(t) = P_0 \left(1 + \frac{r}{k}\right)^{kt}.$$

(See chapter 4 in the textbook for more details).

In this problem, we know $P(4)$, r and k and we want to know P_0:

$$25000 = P(4) = P_0 \left(1 + \frac{0.055}{365}\right)^{1460} \approx 1.246056 P_0 \implies P_0 = 25000/1.246056 \approx 20063.3.$$

2. Use the change of base formula to compute the following logarithms. Do not use a calculator — express your answers in terms of $\ln 2$, $\ln 3$, $\ln 5$, and $\ln 7$. For example

$$\log_7 100 = \frac{\ln 100}{\ln 7} = \frac{\ln 10^2}{\ln 7} = \frac{2\ln 10}{\ln 7} = \frac{2(\ln 2 + \ln 5)}{\ln 7} = \frac{2\ln 2 + 2\ln 5}{\ln 7}.$$

(a) $\log_5 36 = \dfrac{\ln 36}{\ln 5} = \dfrac{2\ln 2 + 2\ln 3}{\ln 5}.$

(b) $\log_{20} 21 = \dfrac{\ln 21}{\ln 20} = \dfrac{\ln 3 + \ln 7}{2\ln 2 + \ln 5}.$

(c) $\log_{10} \sqrt{75} = \dfrac{0.5\ln 75}{\ln 10} = \dfrac{0.5\ln 3 + \ln 5}{\ln 2 + \ln 5}.$

(d) $\log_{21} \dfrac{1}{\sqrt[3]{50}} = \dfrac{-\frac{1}{3}\ln 50}{\ln 21} = -\dfrac{\frac{2}{3}\ln 5 + \frac{1}{3}\ln 2}{\ln 3 + \ln 7}.$

3. Simplify the following expressions using properties of the natural log function.

(a) $\ln\left(\dfrac{x^2 + 3x + 1}{5x + 3}\right) = \ln(x^2 + 3x + 1) - \ln(5x + 3)$

(b) $\ln \sqrt[3]{\dfrac{5xy^3}{x^2 + y^2}} = \frac{1}{3}\left(\ln 5 + \ln x + 3\ln y - \ln\left(x^2 + y^2\right)\right)$

$$= \tfrac{1}{3}\ln 5 + \tfrac{1}{3}\ln x + \ln y - \tfrac{1}{3}\ln(x^2 + y^2).$$

4. Solve the equations.

(a) $3x^2 + 5x - 8 = 0 \implies x = \dfrac{-5 \pm \sqrt{25+96}}{6} = \dfrac{-5 \pm 11}{6} \implies \begin{cases} x_1 &= 1 \\ x_2 &= -8/3 \end{cases}$

(b) $\dfrac{2x+1}{x-2} = \dfrac{3x+5}{8-2x} \implies (2x+1)(8-2x) = (3x+5)(x-2) \implies 7x^2 - 15x - 18 = 0$

$\implies x = \dfrac{15 \pm \sqrt{225+504}}{14} = \dfrac{15 \pm 27}{14} \implies \begin{cases} x_1 &= 3 \\ x_2 &= -6/7 \end{cases}$

5. Solve the pairs of equations.

(a) $\begin{cases} 4x + 5y &= 7 \\ 3x + 4y &= 13 \end{cases}$

First approach:

(i) Subtract $5 \times (3x + 4y = 13)$ from $4 \times (4x + 5y = 7)$:

$$\begin{array}{rcr} 4 \times (4x + 5y & = & 7) \\ -\ 5 \times (3x + 4y & = & 13) \\ \hline x & = & -37 \end{array}$$

(ii) Subtract $3 \times (4x + 5y = 7)$ from $4 \times (3x + 4y = 13)$:

$$\begin{array}{rcr} 4 \times (3x + 4y & = & 13) \\ -\ 3 \times (4x + 5y & = & 7) \\ \hline y & = & 31 \end{array}$$

(iii) Solution: $(x, y) = (-37, 31)$.

Second approach:

(i) Solve $4x + 5y = 7$ for y (or for x): $y = \dfrac{7-4x}{5}$.

(ii) Substitute this into the second equation:

$$3x + 4y = 13 \implies 3x + 4\left(\frac{7-4x}{5}\right) = 13 \implies 15x + 28 - 6x = 65 \implies x = -37,$$

(iii) Find the corresponding y-value:

$$y = \frac{7 - 4(-37)}{5} = \frac{7+148}{5} = \frac{155}{5} = 31.$$

(iv) Solution (as before): $(x, y) = (-37, 31)$.

(b) $\begin{cases} 3x - 2y &= 1 \\ 5x + y &= 2 \end{cases}$

I'll use the first approach for this one:

(i) Add $(3x - 2y = 1)$ to $2 \times (5x + y = 2)$:

$$\begin{array}{rcr} (3x - 2y & = & 1) \\ +2\ (5x + y & = & 2) \\ \hline 13x & = & 5 \end{array} \implies x = \frac{5}{13}$$

(ii) Subtract $3 \times (5x + y = 2)$ from $5 \times (3x - 2y = 1)$:

$$\begin{array}{rcl} 5 \times (3x - 2y & = & 1) \\ -\ 3\ \times (5x + y & = & 2) \\ \hline -13y & = & -1 \end{array} \implies y = \frac{1}{13}$$

(iii) Solution: $(x, y) = (5/13, 1/13)$.

(c) $\begin{cases} x^2 + 2x - 3y & = & -1 \\ 4x + 2y & = & 14 \end{cases}$

I'll use the second approach here:

(i) Solve $4x + 2y = 14$ for y: $y = 7 - 2x$.

(ii) Substitute this into the first equation:

$$x^2 + 2x - 3y = -1 \implies x^2 + 2x - 3(7 - 2x) = -1 \implies x^2 + 8x - 20 = 0$$

$$\implies x = \frac{-8 \pm \sqrt{64 + 80}}{2} = \frac{-8 \pm 12}{2} \implies \begin{cases} x_1 & = & 2 \\ x_2 & = & -10 \end{cases}$$

(iii) Find the corresponding y-values:

$$y_1 = 7 - 2x_1 = 3 \text{ and } y_2 = 7 - 2x_2 = 27.$$

(iv) Solutions: $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-10, 27)$.

6. The demand equation for a monopolistic firm's product is $p = 20 - 0.4q$, where p is the price/unit of the firm's product (measured in dollars) and q is the daily demand for the firm's product, measured in 100s of units.

(a) Find the firm's ***revenue function***, $r = f(q)$, where r is the firm's daily revenue. What kind of function is this (algebraically speaking)? What is its graph?

Revenue is price×quantity, so in this case the revenue function is

$$r = pq \implies r = (20 - 0.4q)q = 20q - 0.4q^2.$$

This is a *quadratic function* and its graph is a *parabola* that opens *downward* (because the coefficient of q^2 is negative).

(b) Find the daily output that ***maximizes*** the firm's daily revenue. You may assume that output = demand.

The maximum revenue is the r-coordinate of the highest point on the graph of the function $r = 20q - 0.4q^2$, which occurs at the *vertex* of the downward opening parabola. We want to find the q-coordinate (the abscissa) of the vertex, which occurs halfway between the two zeros of the revenue function (the points where the graph crosses the q-axis). The zeros occur at the points where $(20 - 0.4q)q = 0$, i.e., $q_1 = 0$ and $q_2 = 20/0.4 = 50$, so the max revenue occurs when output (= demand) is $q^* = 25$. (Observe that since q is measured in 100s of units, maximum revenue occurs when the output is 2500 units.)

(c) Find the price that the firm should set to maximize its daily revenue. What is the firm's maximum daily revenue *in dollars*?

The firm needs to set the price that corresponds (via the demand equation) to the demand $q^* = 25$. I.e., the revenue-maximizing price is

$$p^* = 20 - 0.4q^* = 20 - 0.4 \cdot 25 = 10.$$

The firm's maximum daily revenue is $r^* = p^*q^* = 250$, which is $\$25,000$ because q is measured in 100s of units, so $r = p \times q$ is measured in $\$ \times 100s = \$100s$.

(d) What is the firm's ***break-even*** average daily cost per unit, assuming maximum revenue?

To break even, the firm's daily cost should equal its daily revenue. In terms of averages, this means that the firm's average daily cost per unit should equal the firm's daily revenue per unit, which is $p^* = \$10$. I.e., to break even (assuming max revenue) the firm's average daily cost per unit needs to be $\bar{c} = \$10$.