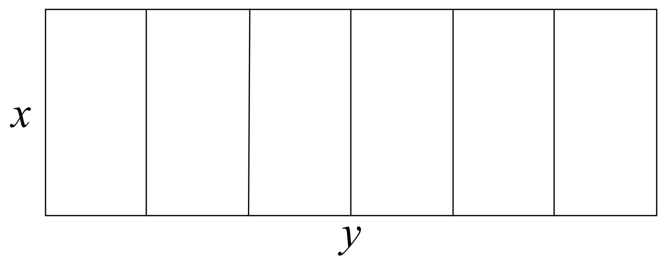
Section 13.6, problem 4.

Label the dimensions of the plot as x (height) and y (width), as illustrated below:



The area of the plot is to be 1400 ft^2, which means that the dimensions must satisfy the condition $xy = 1400$ (measuring x and y in feet). The length of the fencing used is

$$l(x, y) = y + y + x + x + x + x + x + x + x = 2y + 7x.$$

Using the area constraint, $xy = 1400$, we can express the length as a function of x:

$$l(x) = 7x + 2 \cdot \frac{1400}{x} = 7x + \frac{2800}{x}.$$

To find the minimum possible length, we minimize $l(x)$ in the interval $(0, \infty)$ (since length must be positive).

(i) Differentiate: $l'(x) = 7 - \dfrac{2800}{x^2}$

(ii) Find critical point(s) in the interval $(0, \infty)$:

$$l'(x) = 0 \implies 7 = \frac{2800}{x^2} \implies x^2 = \frac{2800}{7} = 400 \implies x = \pm 20.$$

There is only one critical point in $(0, \infty)$, $x^* = 20$.

(iii) Analyze the critical value: $l''(x) = \frac{5600}{x^3}$, so $l''(20) = \frac{5600}{8000} = 0.7 > 0$. This means that $l^* = l(20) = 280$ is a relative minimum value (by the second derivative test), and since there is only one critical point in $(0, \infty)$, l^* is the absolute minimum value in the interval.

(iv) Conclusion: The least length of fence needed is $l^* = 280$ feet.

Comments: (a) You can also solve this problem by expressing the length of fencing as a function of y:

$$l(y) = 2y + 7 \cdot \frac{1400}{y} = 2y + \frac{9600}{y}.$$

(b) You can use the first derivative test instead of the second derivative test to verify that l^* is a minimum value.

Section 13.6, problem 18.

The firm wants to find the rent/apartment, r, that it should charge to maximize its revenue R. We have been given the following information, where we use x to denote the number of \$10 increases in rent:

(a) If $r = 400$, then the firm can rent out all 100 apartments that it owns.

(b) If $r = 400 + 10x$, then the firm can rent out $100 - 2x$ apartments.

As a function of x, the firm's revenue is

$$R(x) = (400 + 10x)(100 - 2x) = 40000 + 200x - 20x^2.$$

The variable x should be nonnegative ($x \geq 0$) because it makes no sense for the firm to lower the rent below \$400, since they only have 100 apartments. On the other hand, x can be no bigger than 50, because the number of apartments that the firm can rent $(100 - 2x)$ cannot be negative.

In other words, we want to maximize the function $R(x) = 40000 + 200x - 20x^2$ in the interval $[0, 50]$ (a closed interval problem).
(i) Differentiate: $R'(x) = 200 - 40x$.

(ii) Find critical point(s) in $[0, 50]$: $R'(x) = 0 \implies 200 = 40x \implies x^* = 5$.

(iii) Analyze critical value: Since this is a closed interval problem, we can simply evaluate $R(x)$ at the endpoints and the critical point — the largest value we see is the maximum we seek.

$$R(0) = 40000, \qquad R(5) = 40500 \quad \text{and} \quad R(50) = 0.$$

(iv) Conclusion: The firm's revenue is maximized when the rent is $r(5) = 400 + 50 = 450$.