ECON/AMS 11A Exam 3 – Solutions SPRING 2018

(1) Consider the function $h(t) = \dfrac{6t-8}{t^2+1}$.

(a) (5 pts) Find the critical points of $h(t)$ and classify the critical values as relative minima, relative maxima or neither.

Differentiate:

$$h'(t) = \frac{6(t^2+1) - 2t(6t-8)}{(t^2+1)^2} = \frac{-6t^2+16t+6}{(t^2+1)^2}$$

Critical point(s):

$$h'(t) = 0 \implies -6t^2+16t+6 = 0 \implies t = \frac{-16 \pm \sqrt{256+144}}{-12} \implies t = \frac{-16 \pm 20}{-12},$$

so there are two critical points: $t_1 = 3$ *and* $t_2 = -1/3$.

First derivative test:

- $h'(-1) = \frac{-16}{4} = -4 < 0$ *and* $h'(0) = 6 > 0$*, so* $h(-1/3) = -9$ *is a relative minimum value.*
- $h'(0) = 6 > 0$ *and* $h'(5) = -64/26^2 > 0$*, so* $h(3) = 1$ *is a relative maximum value.*

(b) (2 pts) Does $h(t)$ have an absolute maximum value or an absolute minimum value in the interval $(0, \infty)$? Justify your answer.

There is an absolute maximum value: $h(3)$ *is a relative maximum value and* $t_1 = 3$ *is only critical point in the interval* $(0,\infty)$*, so* $h(3) = 1$ *is the absolute maximum value of* $h(t)$ *in* $(0,\infty)$.

(2) (6 pts) Find the ***absolute*** maximum and minimum values of the function $f(x) = -2x^3 - 3x^2 + 36x + 15$ in the interval $[0, 5]$. Explain *briefly* how you know that the values you found are in fact the max and min.

This is a closed interval problem — absolute minimum and maximum values are sure to exist and they are sure to occur either at one of the endpoints or at a critical point in the interval. To find these absolute minimum and maximum values...

(a) *Find any critical point(s) of* $f(x)$ *in* $[0, 5]$:

$f'(x) = 6x^2 - 6x + 36$. $f'(x) = 0 \implies 6(-x^2-x+6) = 0 \implies -x^2-x+6 = 0 \implies x = -3$ *or* $x = 2$.

There is only one critical point, $x = 2$*, in the interval* $[0, 5]$.

(b) *Evaluate* $f(x)$ *at both endpoints and at the critical point. The largest value of the three will be the maximum in the interval and the smallest value will be the minimum.*

$$f(0) = 15 \qquad \overbrace{f(2) = 59}^{max} \qquad \overbrace{f(6) = -309}^{min}.$$

(3) (a) Find the quadratic (degree 2) Taylor polynomial for the function $h(x) = \sqrt[4]{x}$, centered at the point $x_0 = 1$.

The quadratic Taylor polynomial for a function $f(x)$ centered at a point x_0 is given by

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

In this case, $f(x) = \sqrt[4]{x} = x^{1/4}$ and $x_0 = 1$, so...

$$f(1) = 1^{1/4} = 1, \quad f'(x) = \frac{1}{4}x^{-3/4} \implies f'(1) = \frac{1}{4} \text{ and } f''(x) = -\frac{3}{16}x^{-7/4} \implies f''(1) = -\frac{3}{16}$$

and the quadratic Taylor polynomial is: $T_2(x) = 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2.$

(b) (2 pts) Use your answer to (a) to estimate $\sqrt[4]{2}$.

$$\sqrt[4]{2} = f(2) \approx T_2(2) = 1 + \frac{1}{4}(2-1) - \frac{3}{32}(2-1)^2 = 1 + \frac{1}{4} - \frac{3}{32} = \frac{37}{32}.$$