Exam 2 –Solutions

Instructions

- There are 5 questions worth a total of 40 points.
- ***For full credit you need to show all your work and explain/justify your answers.***
- You may refer to the textbook, your notes and class notes from the course website.
- You may help each other with the exam but every student needs to write up their own answers and submit their own exam.
- You may use a simple scientific calculator for the computational problems.
- ***Please solve the problems on scratch paper first, clean up your answers and then write the (clean) answers (including explanations and/or work!) neatly in the space provided.***
- **The exam is due in class on Friday, 5/11. You may also turn the exam in on Thursday, 5/10, between 9:00 and 11:30 am to Yonatan's office.**

NAME:____

Problem	Score
1	/8
2	/8
3	/8
4	/8
5	/8
Total	/40

(1) (8 pts) Use the ***rules/formulas*** for differentiation to find derivatives of the functions below. ***Clean up your answers.***

(a) $y = x^3 e^{-0.7x}$

$$\frac{dy}{dx} = \overbrace{3x^2e^{-0.7x} + x^3\underbrace{e^{-0.7x}\cdot(-0.7)}_{\text{chain rule}}}^{\text{product rule}} = \overbrace{3x^2e^{-0.7x} - 0.7x^3e^{-0.7x}}^{\text{clean up}} \qquad (= x^2e^{-0.7x}(3-0.7x))$$

(b) $f(x) = 5\ln\left(x^2\sqrt{4x+3}\right)$

$$= \overbrace{5(\ln x^2 + \ln\sqrt{4x+3}) = 5(2\ln x + \frac{1}{2}\ln(4x+3)) = \underbrace{10\ln x + \frac{5}{2}\ln(4x+3)}_{\text{differentiate this}}}^{\text{simplify first}}$$

$$f'(x) = \frac{10}{x} + \frac{5}{2}\cdot\frac{4}{4x+3} = \overbrace{\frac{10}{x} + \frac{10}{4x+3}}^{\text{clean up}}$$

(c) $v = \dfrac{u^2+3u+1}{5u+4}$

$$v' = \overbrace{\frac{(2u+3)(5u+4) - 5(u^2+3u+1)}{(5u+4)^2}}^{\text{quotient rule}} = \overbrace{\frac{5u^2+8u+7}{(5u+4)^2}}^{\text{clean up}}$$

(d) $g(t) = \sqrt[5]{t^4+3t^2+7}$ $\qquad = \overbrace{\left(t^4+3t^2+7\right)^{1/5}}^{\text{rewrite as a power}}$

$$\frac{dg}{dt} = \overbrace{\frac{1}{5}\left(t^4+3t^2+7\right)^{-4/5}\cdot(4t^3+6t)}^{\text{chain rule}} \qquad \left(= \frac{4t^3+6t}{4\left(t^4+3t^2+7\right)^{4/5}}\right)$$

(2) Consider the function $f(x) = x^{1/3}$.

(a) (4 pts) Find the equation of tangent line to the graph of this function at the point where $x = 1000$.

(b) (4 pts) Use linear approximation to approximate $\sqrt[3]{1003}$. You can use a calculator to check your work, but your answer should be produced without a calculator.

(a) **Point:** $(1000, 1000^{1/3}) = (1000, 10)$.

Slope: $f'(1000) = \frac{d}{dx}(x^{1/3})\Big|_{x=1000} = \frac{1}{3}x^{-2/3}\Big|_{x=1000} = \frac{1}{300}$.

Equation of tangent line: $y - 10 = \frac{1}{300}(x - 1000) \implies y = 10 + \frac{1}{300}(x - 1000)$.

(b) According to linear approximation: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. The function here is $f(x) = x^{1/3}$ and $x = 1003$, so we can apply linear approximation with $x_0 = 1000$:

$$\sqrt[3]{1003} = 1003^{1/3} \approx 1000^{1/3} + \frac{1}{300}(1003 - 1000) = 10 + 0.01 = 10.01.$$

Comment: The linear function $T(x) = f(x_0) + f'(x_0)(x - x_0)$ is the function whose graph is the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. This means that you can approximate $y = f(x)$ by plugging x into the equation of the tangent line from (a).

(3) The ***marginal*** revenue function of a monopolistic firm is given by

$$\frac{dr}{dq} = \frac{3\sqrt{90-q}}{10},$$

where revenue is measured in \$1000s per month and the firm's output q is measured in 1000s of units per month. The firm's production function is

$$q = 18(5l-3)^{1/3},$$

where l is the firm's labor input measured in \$1000s per week.

The firm currently spends \$6000 per week on labor.

(a) (3 pts) Find the firm's output and *marginal product of labor* at the current level of labor input.

(b) (1 pts) Find the firm's ***marginal revenue*** at the current level of labor input.

(c) (1 pts) Find the firm's ***marginal revenue product*** at the current level of labor input.

(d) (3 pts) The firm increases its weekly spending on labor to \$6400. What is the approximate change to the firm's monthly revenue and profit (assuming that all other costs are held fixed)?

(a) Since labor input is measured in \$1000s/week, $l_0 = 6$, and

$$q\Big|_{l=6} = 18(27)^{1/3} = 54 \text{ and } \frac{dq}{dl}\Big|_{l=6} = \frac{1}{3}\cdot 18\cdot(5l-3)^{-2/3}\cdot 5\Big|_{l=6} = 30\cdot 27^{-2/3} = \frac{10}{3}.$$

(b) When $l_0 = 6$, output is $q_0 = 54$ and

$$\frac{dr}{dq}\Big|_{q=54} = \frac{3\sqrt{90-54}}{10} = \frac{9}{5}.$$

(c)

$$\frac{dr}{dl}\Big|_{l=6} = \frac{dr}{dq}\Big|_{q=54}\cdot\frac{dq}{dl}\Big|_{l=6} = \frac{9}{5}\cdot\frac{10}{3} = 6.$$

(d) If labor input increases by \$400, then (i) $\Delta l = 0.4$ (because of the units) and (ii) the firm's monthly *cost* increases by \$1600 (= 4·\$400). To estimate the change in the firm's revenue we use linear approximation:

$$\Delta r \approx \frac{dr}{dl}\Big|_{l=6}\cdot\Delta l = 6\cdot 0.4 = 2.4.$$

I.e., the firm's monthly revenue will increase by about \$2400 (because of the units for revenue), and the firm's monthly profit will increase by about \$800 = \$2400 − \$1600.

(4) The demand equation for a monopolist's product is $q = \sqrt{100 - 0.4p}$.

(a) (2 pts) Find the price-elasticity of demand (as a function of p).

(b) (2 pts) What is the price elasticity of demand when $p = \$200$? Is demand elastic, inelastic, or does demand have unit elasticity at this point?

(c) (2 pts) Suppose that the price is lowered (from \$200) to \$199. Use your answer to part b. to estimate the *percentage* change in demand.

(d) (2 pts) What effect will this change in price have on the firm's revenue? Be as precise as you can, and explain your answer.

(a)

$$\eta_{q/p} = \frac{dq}{dp} \cdot \frac{p}{q} = \overbrace{\frac{1}{2}(100 - 0.4p)^{-1/2} \cdot (-0.4)}^{dq/dp \text{ (chain rule)}} \cdot \frac{p}{\underbrace{(100 - 0.4p)^{1/2}}_{q}} = -\frac{0.2p}{100 - 0.4p}$$

because $(100 - 0.4p)^{-1/2} \cdot \dfrac{1}{(100 - 0.4p)^{1/2}} = \dfrac{1}{(100 - 0.4p)^{1/2}} \cdot \dfrac{1}{(100 - 0.4p)^{1/2}} = \dfrac{1}{100 - 0.4p}$.

(b) Demand is elastic, because

$$\eta_{q/p}\Big|_{p=200} = -\frac{40}{100 - 80} = -2$$

and $|-2| = 2 > 1$.

(c) If the price decreases from \$200 to \$199, then

$$\%\Delta p = \frac{\Delta p}{p} \cdot 100\% = \frac{-1}{200} \cdot 100\% = -0.5\%$$

and using linear approximation,

$$\%\Delta q \approx \eta_{q/p} \cdot \%\Delta p = (-2)(-0.5\%) = 1\%.$$

I.e., demand will *increase* by about 1%.

(d) The marginal revenue is related to the price elasticity of demand by the equation

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right),$$

so when $p = 200$ (and $\eta = -2$) we have

$$\frac{dr}{dq} = 200\left(1 + \frac{1}{-2}\right) = 100 > 0.$$

When the price is decreased from this point, the demand increases (so $\Delta q > 0$) which means that the firm's revenue also increases, because

$$\Delta r \approx \overbrace{\frac{dr}{dq}}^{+} \cdot \overbrace{\Delta q}^{+} > 0.$$

(5) The consumption function for a small country is given by

$$C = \ln\left(\frac{5e^{0.9Y}}{e^{0.1Y}+3}\right),$$

where savings, S, consumption, C, and national income, Y, are all measured in \$ billions.

(a) (1 pt) How much is consumed when $Y = 10$?

(b) (3 pts) What is the marginal propensity to consume when $Y = 10$?

(c) (2 pts) Use your answer to (b) to estimate the change in consumption if income increases from \$10 billion to \$10.4 billion? By how approximately how much will savings change?

(d) (2 pts) Compute the limit $\lim\limits_{Y\to\infty} \frac{dC}{dY}$, and interpret the result.

(*) To make (a) and (b) easier to calculate, we first simplify the consumption function:

$$C = \ln\left(\frac{5e^{0.9Y}}{e^{0.1Y}+3}\right) = \ln 5 + \ln\left(e^{0.9Y}\right) - \ln\left(e^{0.1Y}+3\right) = \ln 5 + 0.9Y - \ln\left(e^{0.1Y}+3\right).$$

(a)

$$C\Big|_{Y=10} = \ln 5 + 9 - \ln\left(e^1+3\right) \approx 8.866$$

i.e., consumption is about \$8.866 billion.

(b) First

$$\frac{dC}{dY} = \frac{d}{dY}\left(\ln 5 + 0.9Y - \ln\left(e^{0.1Y}+3\right)\right) = 0.9 - \frac{0.1e^{0.1Y}}{e^{0.1Y}+3}$$

so

$$\frac{dC}{dY}\Big|_{Y=10} = \left(0.9 - \frac{0.1e^{0.1Y}}{e^{0.1Y}+3}\right)\Big|_{Y=10} = 0.9 - \frac{0.1e}{e+3} \approx 0.852$$

(c) The change in consumption is

$$\Delta C \approx \frac{dC}{dY}\Big|_{Y=10} \cdot \Delta Y \approx 0.852 \cdot 0.4 = 0.341$$

i.e., about \$341 million. and the change in savings is

$$\Delta S = \Delta Y - \Delta C \approx 0.4 - 0.341 = 0.059,$$

or about \$59 million.

(d)

$$\lim_{Y\to\infty}\frac{dC}{dY} = \lim_{Y\to\infty}\left(0.9 - \frac{0.1e^{0.1Y}}{e^{0.1Y}+3}\right) = 0.9 - \lim_{Y\to\infty}\frac{0.1e^{0.1Y}}{e^{0.1Y}+3}$$

$$= 0.9 - \lim_{Y\to\infty}\frac{0.1e^{0.1Y}}{e^{0.1Y}+3}\cdot\overbrace{\left(\frac{e^{-0.1Y}}{e^{-0.1Y}}\right)}^{=1} = 0.9 - \lim_{Y\to\infty}\frac{0.1}{1+3e^{-0.1Y}}$$

$$= 0.9 - \frac{0.1}{\lim\limits_{Y\to\infty} 1+3e^{-0.1Y}} = 0.9 - \frac{0.1}{1} = 0.8,$$

because $\lim\limits_{Y\to\infty} e^{-0.1Y} = 0$.

Interpretation: *As income grows larger, the nation will tend to consume \$0.80 of each additional dollar of income.*