(*) Logarithmic differentiation and $\frac{d}{dx}(e^x)$

In examples (b) and (c) above, we differentiated expressions of the form $\ln(f(x))$ and found that

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}. \tag{1}$$

The expression $f'(x)/f(x)$ is called the *logarithmic derivative* of $f(x)$, for this reason.

As it turns out, there are functions $f(x)$ whose logarithmic derivatives are easier to calculate than their (ordinary) derivatives, because the function $\ln(f(x))$ is easier to differentiate than $f(x)$ itself. In such cases, we can calculate $f'(x)$ by multiplying both sides of the formula (1) by $f(x)$, to find that

$$f'(x) = f(x) \cdot \left(\frac{d}{dx}\ln(f(x))\right).$$

One important application of this idea is to the exponential function $y = e^x$. In this case, $\ln(e^x) = x$, so

$$\frac{d}{dx}\left(e^x\right) = e^x\left(\frac{d}{dx}\ln\left(e^x\right)\right) = e^x\left(\frac{d}{dx}(x)\right) = e^x \cdot 1 = e^x.$$

To reiterate:

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Comment: This property characterizes the exponential function in the sense that if $f'(x) = f(x)$, then $f(x) = ce^x$ for some constant c.

(*) Differentiating a^x.

Logarithmic differentiation works just as easily to find $(a^x)'$ for any fixed $a > 0$. Specifically:

$$\frac{d}{dx}\left(a^x\right) = a^x\left(\frac{d}{dx}\ln\left(a^x\right)\right) = a^x\left(\frac{d}{dx}(x\ln a)\right) = a^x \cdot (\ln a) = (\ln a)a^x.$$

Note that since a is fixed, $\ln a$ is a constant in this context.

(*) Examples — differentiating e^x in combination with other functions.

(e) $\dfrac{d}{dx}\left(2x^3e^x\right) = 6x^2e^x + 2x^3e^x = 2x^2e^x(3+x)$ (product rule).

(f) $\dfrac{d}{dx}e^{5x+3} = e^{5x+3} \cdot (5x+3)' = 5e^{5x+3}$ (chain rule).

(g) $$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{\left(e^x - (-e^{-x})\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x + (-e^{-x})\right)}{\left(e^x + e^{-x}\right)^2} = \dots$$

… using the quotient rule and using the chain rule to find that $(e^{-x})' = e^{-x} \cdot (-1) = -e^{-x}$ …

$$\ldots = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

(*) Percentage rate of change, an example.

A firm's production function is

$$q = 3\,(4l - 5)^{2/3}\,,$$

where q is monthly output and l is monthly labor input, measured in \$1000s.

Current levels of labor input and the firm's output: $l_0 = 8$, so $q_0 = 3(27)^{2/3} = 27$.

If the firm increases its labor input by \$300 a month, then $\Delta l = 0.3$ (because of the units), and linear approximation tells us that

$$\Delta q \approx \left.\frac{dq}{dl}\right|_{l=8} \cdot \Delta l = \left.\left[3 \cdot \frac{2}{3}(4l-5)^{-1/3} \cdot 4\right]\right|_{l=8} \cdot \Delta l = \left.\left[8(4l-5)^{-1/3}\right]\right|_{l=8} \cdot \Delta l = \frac{8}{3} \cdot 0.3 = 0.8.$$

What is the ***percentage*** *change in output here?*

$$\%\Delta q = \frac{\Delta q}{q} \cdot 100\% \approx \frac{\overbrace{0.8}^{\text{linear approx.}}}{27} \cdot 100\% \approx 2.96\%.$$

Rewriting this last approximation (more generally):

$$\%\Delta q = \frac{\Delta q}{q} \cdot 100\% \approx \frac{\overbrace{(dq/dl) \cdot \Delta l}^{\approx \Delta q}}{q} \cdot 100\% = \left(\frac{dq/dl}{q} \cdot 100\%\right) \cdot \Delta l.$$

Terminology: Whereas dq/dl is the *rate of change* of q with respect to l, the expression

$$\left(\frac{dq/dl}{q} \cdot 100\%\right)$$

is called the *percentage rate of change* of q with respect to l.

More generally still, if $y = f(x)$, then $y' = f'(x)$ is the rate of change of y with respect to x, and

$$\frac{y'}{y} \cdot 100\% = \frac{f'(x)}{f(x)} \cdot 100\%$$

is the percentage rate of change of y with respect to x. As we saw in the example above, if Δx is small (close to 0), then linear approximation can be used to approximate the percentage change in y, $\%\Delta y$:

$$\%\Delta y \approx \overbrace{\left(\frac{f'(x)}{f(x)} \cdot 100\%\right)}^{\%\text{-rate of change}} \cdot \Delta x.$$

(*) Elasticity: Please read Supplemental Note # 6 and section 12.3. Again.